Progress on eddy-viscosity models for LES: new differential operators and discretization methods

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European Turbulence Conference 14
Lyon (France), September 1-4 2013
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Progress on eddy-viscosity models for LES
DNS of turbulent incompressible flows

Main features of the DNS code:

- Structured staggered grids
- High-order symmetry-preserving schemes
- Fully-explicit second-order time-integration method
- Poisson solver for 2.5D problems: FFT + PCG
- Hybrid MPI+OpenMP parallelization
- OpenCL-based extension for its use on GPGPU

Air-filled differentially heated cavity at $Ra = 10^{11}$ (111M grid points)

Plane impingement jet at $Re = 20000$ (102M grid points)
DNS of turbulent incompressible flows

Turbulent square duct at $Re_\tau = 1200$ (172M grid points)

Air-filled differentially heated cavity at $Ra = 10^{11}$ (111M grid points)

Square cylinder at $Re = 22000$ (300M grid points)

Plane impingement jet at $Re = 20000$ (102M grid points)

Progress on eddy-viscosity models for LES
Governing equations

Incompressible Navier-Stokes equations:

$$\nabla \cdot u = 0$$
$$\partial_t u + C(u, u) = Du - \nabla p$$

where the **nonlinear convective** term is given by

$$C(u, \phi) = (u \cdot \nabla)\phi$$

and the **linear dissipative** term is given by

$$D\phi = \nu \Delta \phi$$
Stopping the vortex-stretching\textsuperscript{1}

Taking the curl of momentum equation the \textit{vorticity transport equation} follows

\[
\partial_t \omega + C(u, \omega) = C(\omega, u) + D(\omega)
\]

Let us now consider an arbitrary part of the flow domain, \( \Omega \), with \textbf{periodic boundary conditions}. Then, taking the \( L^2 \) innerproduct with \( \omega = \nabla \times u \) leads to the \textit{enstrophy equation}

\[
\frac{1}{2} \frac{d}{dt} (\omega, \omega) = (\omega, C(\omega, u)) - \nu (\nabla \omega, \nabla \omega)
\]

where \( (a, b) = \int_{\Omega} a \cdot b d\Omega \). Unless, the grid is fine enough convection dominates diffusion (in a discrete sense)

\[
(\omega, C(\omega, u)) > \nu (\nabla \omega, \nabla \omega)
\]

\textsuperscript{1}F.X. Trias \textit{et al.} \textbf{Computers\&Fluids}, 39:1815-1831, 2010

Progress on eddy-viscosity models for LES
Stopping the vortex-stretching

The vortex-stretching term can be expressed in terms of the invariant
\[ R = -\frac{1}{3} tr(S^3) = -det(S) \]

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\[ R = \frac{1}{3} tr(S^3) = \frac{1}{3} tr(S^3) = -det(S) \]

Then, recalling that \( \nabla \times \omega = \nabla (\nabla \cdot u) - \Delta u \) and the boundary contribution vanishes \( * \), the diffusive term is given by the \( L^2(\Omega) \)-norm of \( \Delta u \)

\[ (\omega, \mathcal{C}(\omega, u)) = 4 \int_{\Omega} R d\Omega \]

\[ (\nabla \omega, \nabla \omega) = - (\omega, \Delta \omega) = (\omega, \nabla \times \nabla \times \omega) \]

\[ = (\nabla \times \omega, \nabla \times \omega) = (\Delta u, \Delta u) = \| \Delta u \|^2 \]
Stopping the vortex-stretching

The **overall damping** introduced by a model should be given by

\[
H^\Omega = \min \left\{ \frac{\nu \|\Delta u\|^2}{4|\tilde{R}|}, 1 \right\}
\]

where \( \tilde{R} = \int_{\Omega} R d\Omega \).

Notice that any model based on this ratio automatically **switches off** for:

- Laminar flows (\( R \to 0 \))
- 2D flows (\( \lambda_2 = 0 \to R = 0 \))
- In the wall (near-wall behavior is given by \( R \propto y^1 \) and \( \|\Delta u\|^2 \propto y^0 \))
Stopping the vortex-stretching

The overall damping introduced by a model should be given by

\[ H^\Omega = \min \left\{ \frac{\nu \|\Delta u\|^2}{4|\tilde{R}|}, 1 \right\} \]

One possible solution would consist on an eddy-viscosity type LES model:

\[ \nu_t \approx \frac{4|\tilde{R}|}{\|\Delta u\|^2} \]

Taking \( \|\Delta u\|^2 \leq -\lambda_\Delta(\omega, \omega) = 4\lambda_\Delta \tilde{Q} \), it becomes the eddy-viscosity model\(^2\) based on the invariants \( R = -1/3 tr(S^3) = -det(S) \) and \( Q = -1/2 tr(S^2) \).

\( \lambda_\Delta < 0 \) is the largest (smallest in absolute value) non-zero eigenvalue of Laplacian operator \( \Delta \) on \( \Omega \). In a periodic box of size \( h \), \( \lambda_\Delta = -(\pi/h)^2 \).

Stopping the vortex-stretching

The **overall damping** introduced by a model should be given by

\[
H^\Omega = \min \left\{ \frac{\nu \|\Delta u\|^2}{4|\tilde{R}|}, 1 \right\}
\]

Alternatively, **regularizations** of the non-linear convective term results into a damping of vortex-stretching term, *i.e.* \(f^\text{reg} |\tilde{R}|\) (where \(0 < f \leq 1\))

\[
f^\text{reg} \approx \min \left\{ \frac{\nu \|\Delta u\|}{4|\tilde{R}|}, 1 \right\}
\]

Or a combination of both?
Towards a simple LES model

Hence, a new eddy-viscosity model for LES

\[ \partial_t \overline{u} + C(\overline{u}, \overline{u}) = D\overline{u} - \nabla p - \nabla \cdot \tau(\overline{u}) ; \quad \nabla \cdot \overline{u} = 0 \]

\[ \tau(\overline{u}) = -2\nu_t S(\overline{u}) \]

has been derived from the criterion that vortex-stretching mechanism must stop at the smallest grid scale

\[ \nu_t \approx \frac{4|\tilde{R}|}{||\Delta \overline{u}||^2} \]

And what about the implementation?

- No problems with \(4|\tilde{R}|\) and \(||\Delta \overline{u}||^2\).
- But, what about the discretization of \(\nabla \cdot \tau(\overline{u})\)?
Discretization of $2\nabla \cdot (\nu_t S(u))$: a new simple approach\(^3\)

\[
\partial_t u + C(u, u) = Du - \nabla p + 2\nabla \cdot (\nu_t S(u)), \quad \nabla \cdot u = 0
\]
\[
\Omega_s \frac{du_s}{dt} + C(u_s) u_s = Du_s + M^T p_c + ????, \quad M u_s = 0_c
\]

where $2\nabla \cdot (\nu_t S(u)) = \nabla \cdot (\nu_t \nabla u) + \nabla \cdot (\nu_t \nabla u^T)$.

\[
\nabla \cdot (\nu_t \nabla u^T) = \nabla (\nabla \cdot (\nu_t u)) - \nabla \cdot (u \otimes \nabla \nu_t)
\]
\[
= \nabla (\nabla \cdot (\nu_t u)) - C(u, \nabla \nu_t)
\]

\[
\begin{align*}
- M^T \Omega_c^{-1} M \tilde{u}_s & \approx \nabla (\nabla \cdot (\nu_t u)) \\
- C(u_s)(- \Omega_s^{-1} M^T \nu_{t,c}) & \approx C(u, \nabla \nu_t)
\end{align*}
\]

where $[\tilde{u}_s]_f = [\nu_{t,s}]_f [u_s]_f$. \hspace{1cm} Straightforward implementation!!!

\(^{3}\)F.X. Trias et al. A simple approach to discretize the viscous term with spatially varying (eddy-)viscosity Journal of Computational Physics, 253:405-417, 2013
Discretization of $2\nabla \cdot (\nu_t S(u))$: a new simple approach

$4^{th}$-order FVM on a staggered Cartesian grid

\[ \nabla \cdot (\mu \nabla u) \]
\[ \nabla \nabla \cdot (\mu u) \]
\[ C(u, \nabla \mu) \]
\[ \nabla \cdot (\mu \nabla u^T) \]
\[ \nabla \cdot (\mu S(u)) \]

Local truncation error vs. Maximum step-size

$p = 3.89$
Discretization of $2\nabla \cdot (\nu_t S(u))$: a new simple approach

$2^{th}$-order FVM on a collocated unstructured grid
Discretization of $2\nabla \cdot (\nu_t S(u))$: a new simple approach

Let’s make it even easier...

$$\nabla \cdot (\nu_t \nabla u^T) = \nabla (\nabla \cdot (\nu_t u)) - C(u, \nabla \nu_t)$$

$$\begin{align*}
- M^T \Omega_{c}^{-1} M \tilde{u}_s - C(u_s)(-\Omega_s^{-1} M^T \nu_{t,c}) \\
\approx \nabla (\nabla \cdot (\nu_t u)) - C(u, \nabla \nu_t)
\end{align*}$$

Since $\nabla (\nabla \cdot (\nu_t u))$ is a gradient of a scalar field, this term can be absorbed into the pressure, $\pi = p - \nabla \cdot (\nu_t u)$.

Therefore, the only term that needs to be discretized is

$$- C(u_s)(-\Omega_s^{-1} M^T \nu_{t,c}) \approx C(u, \nabla \nu_t)$$
Preliminary results

Turbulent channel flow

\[ Re_T = 590 \]

DNS Moser et al. \ LES 64^3

Progress on eddy-viscosity models for LES
Preliminary results

Turbulent square duct

\[ Re_{\tau} = 300 \]

LES 64 × 32 × 32

mean velocity

rms fluctuations

Progress on eddy-viscosity models for LES
Conclusions and Future Research

- The ratio between the invariant $R$ and the (total) dissipation provides a proper differential operator for turbulence models.
- Based on this, a new eddy-viscosity type LES models has been derived.
- A simple new approach to discretize the viscous term for eddy-viscosity models has been proposed.

- Test the performance of new eddy-viscosity type LES for other configurations.
- Try to properly combine regularization modeling and LES.
Thank you for your attention
Further reading

