WORKSHOP
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BUILDING PROPER INVARIANTS FOR LARGE-EDDY SIMULATION

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INTRODUCTION

We consider the numerical simulation of the incompressible Navier-Stokes (NS) equations. In primitive variables they read

$$\partial_t u + \mathcal{C}(u,u) = \mathcal{D}u - \nabla p, \qquad \nabla \cdot u = 0, \tag{1}$$

where u denotes the velocity field, p represents the kinematic pressure, the non-linear convective term is given by $\mathcal{C}(u,v) = (u \cdot \nabla)\, v$, and the diffusive term reads $\mathcal{D}u = \nu \Delta u$, where ν is the kinematic viscosity. Direct simulations at high Reynolds numbers are not feasible because the convective term produces far too many scales of motion. Hence, in the foreseeable future, numerical simulations of turbulent flows will have to resort to models of the small scales. The most popular example thereof is the Large-Eddy Simulation (LES). Shortly, LES equations result from filtering the NS Eqs.(1) in space

$$\partial_t \overline{u} + \mathcal{C}(\overline{u},\overline{u}) = \mathcal{D}\overline{u} - \nabla \overline{p} - \nabla \cdot \tau(\overline{u}) \ ; \qquad \nabla \cdot \overline{u} = 0, \tag{2}$$

where $\overline{u}$ is the filtered velocity and $\tau(\overline{u})$ is the subgrid stress tensor and aims to approximate the effect of the under-resolved scales, *i.e.* $\tau(\overline{u}) \approx \overline{u \otimes u} - \overline{u} \otimes \overline{u}$. Because of its inherent simplicity and robustness, the eddy-viscosity assumption is by far the most used closure model

$$\tau(\overline{u}) \approx -2\nu_e S(\overline{u}), \tag{3}$$

where ν_e denotes the eddy-viscosity. Following the same notation than in [1], the eddy-viscosity can be modeled as follows

$$\nu_e = (C_m \delta)^2 D_m(\overline{u}), \tag{4}$$

where δ is a subgrid characteristic length. C_m and D_m are the constant and differential operator associated with the model.

A 5D PHASE SPACE FOR EDDY-VISCOSITY MODELS

The essence of turbulence are the smallest scales of motion. They result from a subtle balance between convective transport and diffusive dissipation. Numerically, if the grid is not fine enough, this balance needs to be restored by a turbulence model. The success of a turbulence model depends on the ability to capture well this (im)balance. In this regard, many eddy-viscosity models for LES have been proposed in the last decades (see [3], for a review). In order to be frame invariant, most of them rely on differential operators that are based on the combination of invariants of a symmetric second-order tensor (with the proper scaling factors). To make them locally dependent such tensors are derived from the gradient of the resolved velocity field, $G \equiv \nabla \overline{u}$. This is a second-order traceless tensor, $tr(G) = \nabla \cdot \overline{u} = 0$. Therefore, it contains 8 independent elements and it can be characterized by 5 invariants (3 scalars are required to specify the orientation in 3D). Following the same criterion as in [4], this set of five invariants can be defined as follows

Invariants

Q_G	R_G	Q_S	R_S	V^2	Q_Ω	Z^2
$\mathcal{O}(y^2)$	$\mathcal{O}(y^3)$	$\mathcal{O}(y^0)$	$\mathcal{O}(y^1)$	$\mathcal{O}(y^0)$	$\mathcal{O}(y^0)$	$\mathcal{O}(y^2)$
$[T^{-2}]$	$[T^{-3}]$	$[T^{-2}]$	$[T^{-3}]$	$[T^{-4}]$	$[T^{-2}]$	$[T^{-4}]$

Models

Smagorinsky	WALE	Vreman's	R_S-based	σ-model
Eq.(6)	Eq.(7)	Eq.(7)	Ref. [2]	Ref. [1]
$\mathcal{O}(y^0)$	$\mathcal{O}(y^3)$	$\mathcal{O}(y^1)$	$\mathcal{O}(y^1)$	$\mathcal{O}(y^3)$

Table 1: Top: near-wall behavior and units of the five basic invariants in the 5D phase space given in (5) together with the invariants $Q_\Omega = Q_G - Q_S$ and $Z^2 = V^2 - 2Q_S Q_\Omega$. Bottom: near-wall behavior of the Smagorinsky, the WALE, the Vreman's, the R_S-based and the σ-models.

$$\{Q_G, R_G, Q_S, R_S, V^2\}, \tag{5}$$

where $Q_A = 1/2\{tr^2(A) - tr(A^2)\}$ and $R_A = det(A)$ represent the second and third invariants of the second-order tensor A, respectively. Moreover, the first invariant of A is denoted as $P_A = tr(A)$. Finally, $V^2 = tr(S^2\Omega^2)$, where $S = 1/2(G + G^T)$ and $\Omega = 1/2(G - G^T)$ are the symmetric and the skew-symmetric parts of the gradient tensor, G. Starting from the classical Smagorinsky model [5] that reads

$$\nu_e^{Smag} = (C_S \delta)^2 |S(\overline{u})| = 2(C_S\delta)^2(-Q_S)^{1/2}, \tag{6}$$

most of the eddy-viscosity models for LES are based on invariants of second-order tensors that are derived from the gradient tensor, G. Therefore, it seems natural to re-write them in terms of the 5D phase space defined in (5). For instance, the WALE [6] and the Vreman's model [7] read

$$\nu_e^W = (C_W \delta)^2 \frac{(2/3Q_G^2 + Z^2)^{3/2}}{(-2Q_S)^{5/2} + (2/3Q_G^2 + Z^2)^{5/4}}, \tag{7}$$

$$\nu_e^{Vr} = (C_{Vr}\delta)^2 \left(\frac{Q_G^2 + 4Z^2}{2(Q_\Omega - Q_S)} \right)^{1/2}, \tag{8}$$

respectively, where $Q_\Omega = Q_G - Q_S$ and $Z^2 = V^2 - 2Q_S Q_\Omega$. Other eddy-viscosity models that can be re-written in terms of the above-defined invariants are the model proposed by

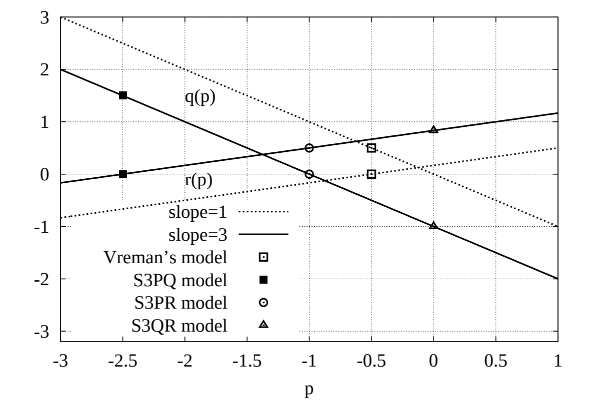

Figure 1: Solutions for the linear system of Eqs.(9) for $s = 1$ (dashed line) and $s = 3$ (solid line). Each (r, q, p) solution represents an eddy-viscosity model of the form given in Eq.(9).

Verstappen [2] and the σ-model proposed in [1]. The major drawback of the Smagorinsky model is that the differential operator it is based on does not vanish in near-wall regions (see Table 1). It is possible to build models based on invariants that do not have this limitation. Examples thereof are the WALE, the Vreman's, the Verstappen's and the σ-model.

BUILDING NEW PROPER INVARIANTS FOR LES MODELS

At this point, it is interesting to observe that new models can be derived by imposing restrictions on the differential operators they are based on. For instance, let us consider models that are based on the invariants of the tensor GG^T

$$\nu_e = (C_M \delta)^2 P_{GG^T}^p Q_{GG^T}^q R_{GG^T}^r, \tag{9}$$

where $-6r - 4q - 2p = -1$, $6r + 2q = s$ and $P_{GG^T} = 2(Q_\Omega - Q_S)$, $Q_{GG^T} = Q_G^2 + 4Z^2$ and $R_{GG^T} = R_G^2$, respectively. The above-defined restrictions on the exponents follow by imposing the $[T^{-1}]$ units of the differential operator and the slope, s, for the asymptotic near-wall behavior (see Table 1), *i.e.* $\mathcal{O}(y^s)$. Solutions for $q(p, s) = (1 - s)/2 - p$ and $r(p, s) = (2s - 1)/6 + p/3$ are displayed in Figure 1. The Vreman's model given in Eq.(7) corresponds to the solution with $s = 1$ (see Table 1) and $r = 0$. However, it seems more appropriate to look for solutions with the proper near-wall behavior, *i.e.* $s = 3$ (solid lines in Figure 1). Restricting to solutions involving only two invariants of GG^T we find three new models (see Figure 1),

$$\nu_e^{S3PQ} = (C_{s3pq}\delta)^2 P_{GG^T}^{-5/2} Q_{GG^T}^{3/2}, \tag{10}$$

$$\nu_e^{S3PR} = (C_{s3pr}\delta)^2 P_{GG^T}^{-1} R_{GG^T}^{1/2}, \tag{11}$$

$$\nu_e^{S3QR} = (C_{s3qr}\delta)^2 Q_{GG^T}^{-1} R_{GG^T}^{5/6}, \tag{12}$$

where the model constants, C_{s3xx}, can be related with the Vreman's constant, C_{Vr}, with the following inequality

$$0 \le \frac{(C_{Vr})^2}{(C_{s3xx})^2} \frac{\nu_e^{S3xx}}{\nu_e^{Vr}} \le \frac{1}{3}. \tag{13}$$

Hence, imposing $C_{s3pq} = C_{s3pr} = C_{s3qr} = \sqrt{3} C_{Vr}$ guarantees both numerical stability and that the models have less or equal dissipation than Vreman's model, *i.e.*

$$0 \le \nu_e^{S3xx} \le \nu_e^{Vr}. \tag{14}$$

Figure 2 shows the performance of the proposed models for a turbulent channel flow in conjunction with the discretization methods for eddy-viscosity models proposed in [8]. Compared with Vreman's model, they improve the results near the wall.

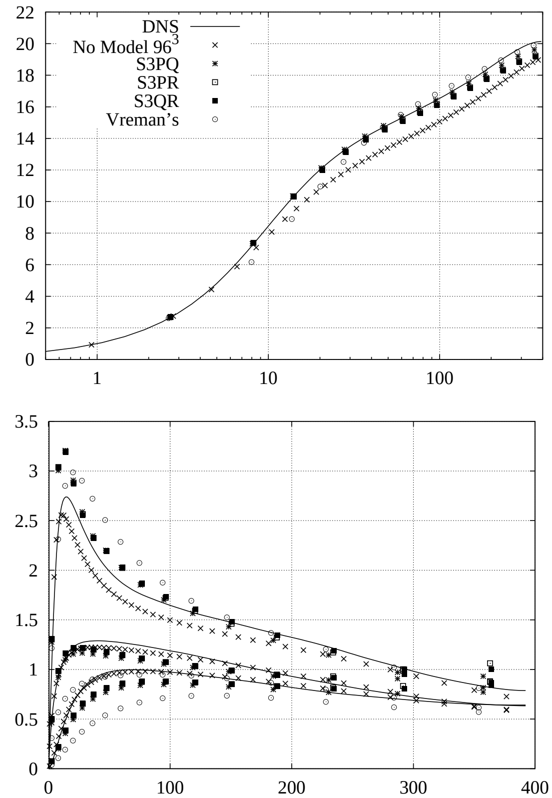

Figure 2: Results for a turbulent channel flow at $Re_\tau = 395$ obtained with a 32^3 mesh for LES and a 96^3 mesh without model, *i.e.* $\nu_e = 0$. Solid line corresponds to the DNS by Moser *et al.* [9].

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