Building proper invariants for large-eddy simulation

F.X. Trias, D. Folch, A. Gorobets, and A. Oliva

1 Introduction

We consider the simulation of the incompressible Navier-Stokes (NS) equations

$$\partial_t u + (u \cdot \nabla) u = \nu \nabla^2 u - \nabla p, \qquad \nabla \cdot u = 0, \tag{1}$$

where u denotes the velocity field, p represents the kinematic pressure and ν is the kinematic viscosity. In the foreseeable future, numerical simulations of turbulent flows will have to resort to models of the small scales because the non-linear convective term produces too many scales of motion. Large-Eddy Simulation (LES) is probably the most popular example thereof. In short, LES equations result from applying a spatial filter, with filter length Δ, to the NS Eqs.(1)

$$\partial_t \overline{u} + (\overline{u} \cdot \nabla) \overline{u} = \nu \nabla^2 \overline{u} - \nabla \overline{p} - \nabla \cdot \tau(\overline{u}), \qquad \nabla \cdot \overline{u} = 0, \tag{2}$$

where $\overline{u}$ is the filtered velocity and $\tau(\overline{u})$ is the subgrid stress (SGS) tensor and aims to approximate the effect of the under-resolved scales, *i.e.* $\tau(\overline{u}) \approx \overline{u \otimes u} - \overline{u} \otimes \overline{u}$. In this regard, the eddy-viscosity assumption is by far the most used closure model

$$\tau(\overline{u}) \approx -2\nu_e S(\overline{u}), \tag{3}$$

where ν_e is the eddy-viscosity. Following [1], the it can be modeled as follows

$$\nu_e = (C_m \Delta)^2 D_m(\overline{u}), \tag{4}$$

F.X. Trias · D. Folch · A. Gorobets · A. Oliva
Heat and Mass Transfer Technological Center, Technical University of Catalonia, ETSEIAT, C/Colom 11, 08222 Terrassa, Spain, e-mail: {xavi,davidf,andrey,oliva}@cttc.upc.edu

A.Gorobets
Keldysh Institute of Applied Mathematics, 4A, Miusskaya Sq., Moscow 125047, Russia, e-mail: cherepock@gmail.com

Invariants					
Q_G	R_G	Q_S	R_S	V^2	Q_Ω
$\mathscr{O}(y^2)$	$\mathscr{O}(y^3)$	$\mathscr{O}(y^0)$	$\mathscr{O}(y^1)$	$\mathscr{O}(y^0)$	$\mathscr{O}(y^0)$
$[T^{-2}]$	$[T^{-3}]$	$[T^{-2}]$	$[T^{-3}]$	$[T^{-4}]$	$[T^{-2}]$

Models				
Smagorinsky Eq.(6)	WALE Eq.(7)	Vreman's Eq.(7)	Verstappen's Ref. [2]	σ-model Ref. [1]
$\mathscr{O}(y^0)$	$\mathscr{O}(y^3)$	$\mathscr{O}(y^1)$	$\mathscr{O}(y^1)$	$\mathscr{O}(y^3)$

Table 1 Top: near-wall behavior and units of the five basic invariants in the 5D phase space given in (5) together with the invariant $Q_\Omega = Q_G - Q_S$. Bottom: near-wall behavior of the Smagorinsky, the WALE, the Vreman's, the Verstappen's and the σ-model.

where Δ is a subgrid characteristic length. C_m and D_m are the constant and differential operator associated with the model.

2 A 5D phase space for eddy-viscosity models

The essence of turbulence are the smallest scales of motion. They result from a subtle balance between convective transport and diffusive dissipation. Numerically, if the mesh is too coarse, this balance needs to be restored by a turbulence model. Therefore, the performance of these models strongly depends on the ability to capture well this (im)balance. In this respect, many eddy-viscosity models for LES have been proposed (see [3], for a review). In order to be frame invariant, most of them rely on differential operators that are based on invariants of a symmetric second-order tensor (with the proper scaling factors). Such tensors are usually derived from the gradient of the resolved velocity field, $G \equiv \nabla \overline{u}$; therefore, they are locally dependent and Galilean invariant. This is a second-order traceless tensor, $tr(G) = \nabla \cdot \overline{u} = 0$. Hence, it contains 8 independent elements and it can be characterized by 5 invariants (3 scalars are required to specify the orientation in 3D). Following the same notation as in [4], this set of invariants is given by

$$\{Q_G, R_G, Q_S, R_S, V^2\}, \tag{5}$$

where $Q_A = 1/2\{tr^2(A) - tr(A^2)\}$ and $R_A = det(A)$ are the second and third invariants of the second-order tensor A. The first invariant of A is denoted as $P_A = tr(A)$. Finally, $V^2 = 4(tr(S^2\Omega^2) - 2Q_S Q_\Omega)$, where $S = 1/2(G + G^T)$ is the symmetric part and $\Omega = 1/2(G - G^T)$ is the skew-symmetric part of the tensor G. Going back to the Smagorinsky model [5]

$$\nu_e^{Smag} = (C_S\Delta)^2 |S(\overline{u})| = 2(C_S\Delta)^2(-Q_S)^{1/2}, \tag{6}$$

almost all the eddy-viscosity models for LES are based on invariants of second-order tensors derived from the gradient tensor, G. Hence, they can be re-write in

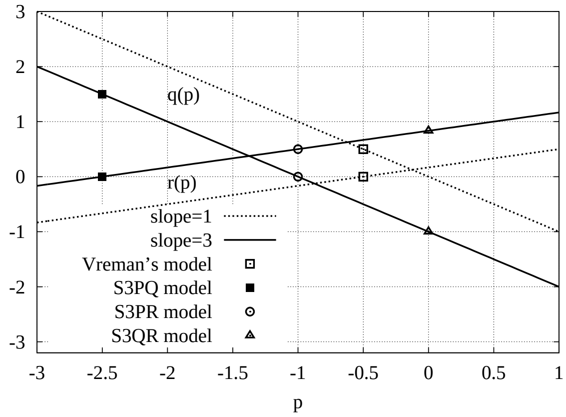

Fig. 1 Solutions for the linear system of Eqs.(9) for $s=1$ (dashed line) and $s=3$ (solid line). Each (r,q,p) solution represents an eddy-viscosity model of the form given in Eq.(9).

terms of the 5D phase space defined in (5). For example, the WALE [6] and the Vreman's model [7] read

$$\nu_e^W = (C_W \Delta)^2 \frac{(V^2/2+2Q_G^2/3)^{3/2}}{(-2Q_S)^{5/2}+(V^2/2+2Q_G^2/3)^{5/4}}, \tag{7}$$

$$\nu_e^{Vr} = (C_{Vr}\Delta)^2 \left(\frac{V^2+Q_G^2}{2(Q_\Omega - Q_S)}\right)^{1/2}. \tag{8}$$

respectively, where $Q_\Omega = Q_G - Q_S$. Other eddy-viscosity models that can be rewritten in terms of the above-defined invariants are the σ-model proposed in [1] and the model proposed by Verstappen [2]. The major drawback of the Smagorinsky model is that the invariant Q_S (see Eq.6) does not vanish in near-wall regions (see Table 1). Nevertheless, it is possible to build models that do not have this limitation: the WALE, the Vreman's, the Verstappen's and the σ-model are examples thereof.

3 Building new proper invariants for LES models

In this context, it is interesting to observe that new models can be derived by imposing appropriate restrictions on the differential operator. For example, we can consider models that are based on the invariants of the tensor GG^T. Namely,

$$\nu_e = (C_M \Delta)^2 P_{GG^T}^p Q_{GG^T}^q R_{GG^T}^r, \tag{9}$$

where $-6r-4q-2p=-1$, $6r+2q=s$ and $P_{GG^T} = 2(Q_\Omega - Q_S)$, $Q_{GG^T} = V^2 + Q_G^2$ and $R_{GG^T} = R_G^2$, respectively. The GG^T tensor appears in the leading term (gradient model) of the Taylor series expansion of the SGS tensor $\tau(\overline{u}) = (\Delta^2/12)GG^T + \mathscr{O}(\Delta^4)$. The above-defined restrictions on the exponents follow by imposing the

$[T^{-1}]$ units of the differential operator and the slope, s, for the asymptotic near-wall behavior (see Table 1), *i.e.* $\mathcal{O}(y^s)$. Solutions for $q(p,s) = (1-s)/2 - p$ and $r(p,s) = (2s-1)/6 + p/3$ are displayed in Figure 1. The Vreman's model given in Eq.(7) corresponds to the solution with $s = 1$ (see Table 1) and $r = 0$. Nevertheless, solutions with the proper near-wall behavior, *i.e.* $s = 3$ (solid lines in Figure 1) seem more appropriate. Then, restricting ourselves to solutions involving only two invariants of GG^T we find three new eddy-viscosity models (see Figure 1),

$$\nu_e^{S3PQ} = (C_{s3pq}\Delta)^2 P_{GG^T}^{-5/2} Q_{GG^T}^{3/2}, \tag{10}$$

$$\nu_e^{S3PR} = (C_{s3pr}\Delta)^2 P_{GG^T}^{-1} R_{GG^T}^{1/2}, \tag{11}$$

$$\nu_e^{S3QR} = (C_{s3qr}\Delta)^2 Q_{GG^T}^{-1} R_{GG^T}^{5/6}. \tag{12}$$

Lower bounds for the model constants, C_{s3xx}, can be with found (see details in [8])

$$0 \leq \frac{(C_{Vr})^2}{(C_{s3xx})^2} \frac{\nu_e^{S3xx}}{\nu_e^{Vr}} \leq \frac{1}{3}, \tag{13}$$

where C_{Vr} is the Vreman's constant. Hence, imposing $C_{s3pq} = C_{s3pr} = C_{s3qr} = \sqrt{3}C_{Vr}$ guarantees both numerical stability and that the models have less or equal dissipation than Vreman's model, *i.e.*

$$0 \leq \nu_e^{S3xx} \leq \nu_e^{Vr}. \tag{14}$$

4 Results

Figure 2 shows the performance of the proposed models for a turbulent channel flow at $Re_\tau = 395$ together with the discretization methods for eddy-viscosity models proposed in [10]. The code is based on a fourth-order symmetry-preserving discretization finite volume discretization. Results are in good agreement with the DNS data [9]. To illustrate the contribution of the eddy-viscosity models to improve the quality of the solution, the results obtained with a 96^3 mesh without model, *i.e.* $\nu_e = 0$, are also shown. The performance of the three models proposed here (S3PQ, S3PR and S3QR) is essentially the same. Compared with the Vreman's model, they tend to improve the results for the mean velocity in the buffer layer region ($5 < y^+ < 30$) whereas the quality of the solutions in the outer layer ($y^+ > 50$) is very similar. Although some discrepancies are observed, the root-mean-square of the fluctuating velocity components (see Figure 2, bottom) are also in rather good agreement with the DNS data. In this case, the proposed models outperform the solution obtained with the Vreman's model. The latter does not predict accurately the position of the peak for u_{rms}, and clearly under-predict the solution for both v_{rms} and w_{rms}. In this case, the solution obtained without model may seem accurate; however, the clearly over-predicted friction velocity (if results were normalized by

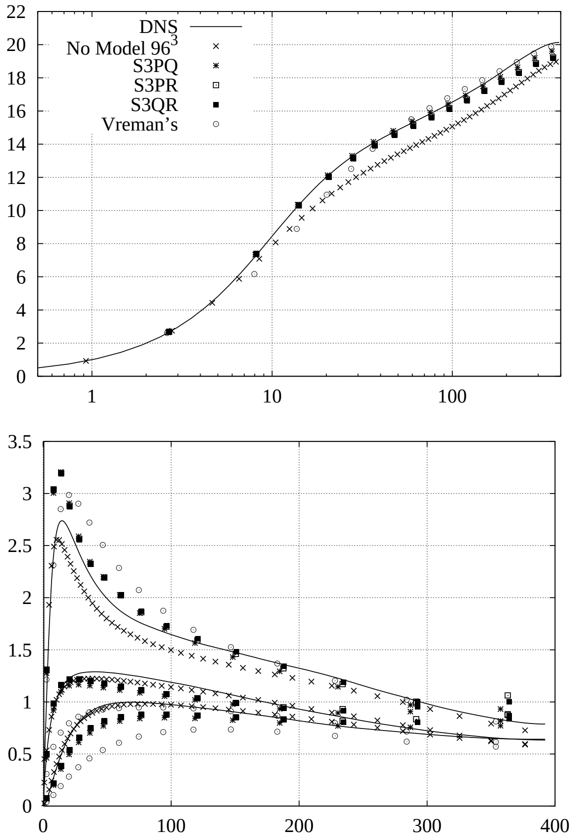

Fig. 2 Results for a turbulent channel flow at $Re_\tau = 395$ obtained with a 32^3 mesh for LES and a 96^3 mesh without model, *i.e.* $\nu_e = 0$. Solid line corresponds to the DNS by Moser *et al.* [9]. Top: average stream-wise velocity, $\langle u \rangle$. Bottom: root-mean-square of the fluctuating velocity components (from top to bottom, u_{rms}, w_{rms} and v_{rms}, respectively.)

the mean stream-wise velocity) compensates an over-prediction of the velocity fluctuations. These results support the idea that the Vreman's model tends to dissipate too much in the near-wall region where the eddy-viscosity, ν_e, does not follow the proper cubic behavior (see Table 1). To illustrate this, the average eddy-viscosity, $\langle \nu_e \rangle$, divided by the kinematic viscosity, ν, is displayed in Figure 3. Results using the classical Smagorinsky model are also shown for comparison. As expected, the proposed models follow a cubic near-wall behavior whereas the Vreman's model predict much higher values in the buffer layer region ($5 < y^+ < 30$). Our current research is focused on finding a proper definition of the subgrid characteristic length, Δ which is also a key element of any eddy-viscosity model (see Eq. 4). Preliminary results in this regard can be found in [11].

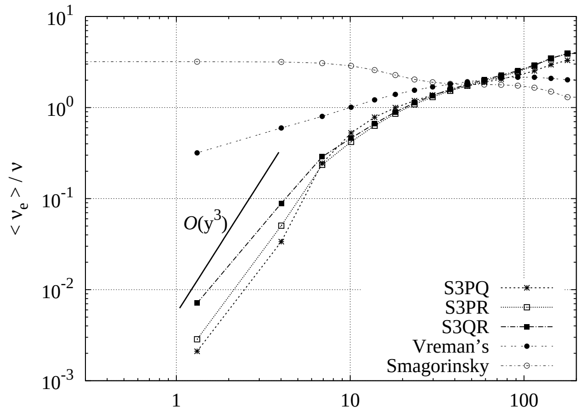

Fig. 3 Averaged eddy-viscosity, $\langle \nu_e \rangle$, divided by the kinematic viscosity, ν. Results for a turbulent channel flow at $Re_\tau = 395$ obtained with a 32^3 for different LES models.

Acknowledgements This work has been financially supported by the *Ministerio de Economía y Competitividad*, Spain (ENE2014-60577-R), and a *Ramón y Cajal* postdoctoral contract (RYC-2012-11996). Calculations have been performed on the IBM MareNostrum supercomputer at the Barcelona Supercomputing Center. The authors thankfully acknowledge these institutions.

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