A new subgrid characteristic length for LES

F.X. Trias, A. Gorobets, and A. Oliva

1 Introduction

Large-eddy simulation (LES) equations result from applying a spatial commutative filter, with filter length Δ, to the Navier-Stokes equations

$$\partial_t \overline{u} + (\overline{u} \cdot \nabla) \overline{u} = \nu \nabla^2 \overline{u} - \nabla \overline{p} - \nabla \cdot \tau(\overline{u}), \qquad \nabla \cdot \overline{u} = 0, \tag{1}$$

where $\overline{u}$ is the filtered velocity and $\tau(\overline{u})$ is the subgrid stress (SGS) tensor and aims to approximate the effect of the under-resolved scales, *i.e.* $\tau(\overline{u}) \approx \overline{u \otimes u} - \overline{u} \otimes \overline{u}$. Most of the difficulties in LES are associated with the presence of walls where SGS activity tends to vanish. Therefore, apart from many other relevant properties, LES models should properly capture this feature [1]. Numerically, this implies an accurate resolution of the near-wall region which results on a high computational cost at high Reynolds numbers. Accurate estimations of these costs, including the temporal scales, are given in the next section. They lead to the conclusion that, in the near future, the feasibility of wall-resolved LES (WRLES) at high-Reynolds numbers should rely on substantial cost reductions in the viscous wall region. This may be achieved by decreasing the number of grid points using high-order schemes or/and using larger time-steps (implicit-explicit time-integration?). Furthermore, it is also concluded that the mesh anisotropy increases with the Reynolds numbers. This represents an additional challenge for WRLES. In this context, a novel definition of subgrid characteristic length, Δ, is proposed with the aim to answer the following research question: *can we find a simple and robust definition of Δ that minimizes the effect of mesh anisotropies on the performance of SGS models?*

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2 Wall-resolved LES: computational costs and mesh anisotropies

In his 1979 pioneering paper, Chapman [2] estimated the number of grid points for an LES of turbulent boundary layers with and without wall modeling as

$$N_{wm} \sim Re_{L_x}^{2/5} \quad \text{and} \quad N_{wr} \sim Re_{L_x}^{9/5}, \tag{2}$$

respectively, where $Re_{L_x} = UL_x/\nu$ is the Reynolds number based on the free-stream velocity, U, and the flat plate length in the streamwise direction, L_x. To reach these scalings, Chapman used the following skin friction correlation

$$c_f = 0.045 Re_\delta^{-1/4}, \tag{3}$$

where $Re_\delta = U\delta/\nu$ is the Reynolds number based on the boundary layer thickness, $\delta(x)$, and assumed a seventh-power velocity distribution law, *i.e.* $u \sim y^{1/7}$. The later leads to an exact relation between the momemtum thickness, θ, and δ given by $\theta = 7\delta/72$. Then, using Eq.(3) and $c_f = 2d\theta/dx$ leads to

$$\frac{\delta}{x} = 0.37 Re_x^{-1/5} \quad \text{and} \quad c_f = 0.0577 Re_x^{-1/5}, \tag{4}$$

where $Re_x = Ux/\nu$ is the Reynolds number based on the streamwise distance from the leading edge, x. From these equations it is relatively easy to show the scaling given by Chapman in Eq.(2). Recently, Choi and Moin [3] gave new estimations based on a more accurate skin friction correlation for high Reynolds numbers ($10^6 \leq Re_x \leq 10^9$) given by

$$c_f = 0.020 Re_\delta^{-1/6}. \tag{5}$$

In this case, the analysis leads to

$$N_{wm} \sim Re_{L_x} \quad \text{and} \quad N_{wr} \sim Re_{L_x}^{13/7}. \tag{6}$$

These findings are extensively used to emphasize the prohibitive costs of LES without wall-modeling and the necessity, in the foreseeable future, of wall-modeling techniques for applications at high Reynolds numbers. However, under some assumptions, these scalings are only valid for a range of Re_x; moreover, they do not include the costs associated with temporal scales which eventually can be even more restrictive due to the inherent difficulty (impossibility?) to parallelize LES equations in time. These two issues are addressed in the next paragraphs. Let us consider a general power-law for the skin friction coefficient

$$c_f = a Re_\delta^\beta. \tag{7}$$

Then, following the above explained reasonings it leads to

$$\frac{\delta}{x} = b Re_x^\alpha \quad \text{and} \quad c_f = 7b/36(\alpha+1) Re_x^\alpha, \tag{8}$$

where $b = (36a(1-\beta)/7)^{1/(1-\beta)}$ and $\alpha = \beta/(1-\beta)$. Notice that with $a = 0.045$ and $\beta = -1/4$ it leads to the Chapman's scalings given in Eqs.(4). Following the same reasonings as in Ref.[3] the number of grid points in the outer layer and the viscous wall region can be estimated as follows

$$N^{out} = n_x n_y n_z \left(\frac{1}{b^2(1+2\alpha)}\right) \frac{L_z}{L_x} Re_{L_x}^{-2\alpha} \left(\left(\frac{Re_{L_x}}{Re_{x_0}}\right)^{1+2\alpha} - 1\right), \tag{9}$$

$$N^{vis} = \frac{n_y^w}{\Delta x_w^+ \Delta z_w^+} \frac{7b}{72} \frac{L_z}{L_x} Re_{L_x}^{2+\alpha} \left(1 - \left(\frac{Re_{x_0}}{Re_{L_x}}\right)^{1+\alpha}\right), \tag{10}$$

where $n_x n_y n_z$ is the number of grid points to resolve the cubic volume $\delta^3(x)$ in the outer layer (typically in the range $10^3 - 10^4$ [3]), L_z is the spanwise length and x_0 is the initial streamwise location where the skin friction correlation (8) holds. Then, Δx_w^+, Δz_w^+ and n_y^w are respectively the grid resolutions (in wall units) and the number of grid points in the wall-normal direction in the viscous wall region, *i.e.* $0 \leq y^+ \lesssim l_y^+ \approx 100$. Typical values for WRLES lead to $n_y^w/(\Delta x_w^+ \Delta z_w^+) \sim 0.01$ [3]. This analysis can be extended giving estimations of the number of time-steps for the outer layer and the viscous wall region

$$N_t^{out} = \frac{N_{TU} n_x}{b C_{conv}} Re_{L_x} Re_{x_0}^{-(1+\alpha)}; \qquad N_t^{vis} = \max(N_{t_{diff}}^{vis}, N_{t_{conv}}^{vis}), \tag{11}$$

where

$$N_{t_{diff}}^{vis} = \frac{N_{TU}}{C_{diff}} \frac{7b}{72} \frac{\alpha+1}{(\Delta y_w^+)^2} Re_{L_x} Re_{x_0}^{\alpha}; \qquad N_{t_{conv}}^{vis} = \frac{N_{TU}}{C_{conv}} \sqrt{\frac{7b}{72} \frac{\alpha+1}{(\Delta x_w^+)^2}} Re_{L_x} Re_{x_0}^{\alpha/2}, \tag{12}$$

where N_{TU} is the number of time-units, L_x/U, to be computed; C_{conv} and C_{diff} are the convective and diffusive constants in the CFL condition. In summary, combining Eqs.(9), (10) and (11) leads to the following costs for LES with and without wall-modeling:

$$N_t^{wm} N_{wm} \sim Re_{L_x}^2 \quad \text{and} \quad N_t^{wr} N_{wr} \sim Re_{L_x}^{3+\alpha}. \tag{13}$$

Nowadays, this represents the main limitation of (wall-resolved) LES. On the other hand, it is also possible to give estimations of the mesh anisotropy, *i.e.* $\Delta x/\Delta y$, in the boundary layer. Namely, in the viscous sublayer, $\max(\Delta x/\Delta y) = \Delta x_w^+/\Delta y_w^+ \approx 50 - 100$ is not expected to change with the Reynolds number. However, in the overlap region ($y^+ \gtrsim 50$, $y/\delta < 0.1$) where control volumes of the viscous wall region and the outer layer ($y^+ \gtrsim 50$) must be smoothly connected, the grid anisotropy can be estimated as

$$\left(\frac{\Delta x}{\Delta y}\right)_{overlap} \approx \frac{(\Delta x)_{out}}{(\Delta y)_{vis}} = \frac{\delta}{n_x} \frac{n_y^w}{l_y}, \tag{14}$$

where l_y is the size of the viscous wall region, *i.e.* $l_y^+ = u_\tau l_y/\nu \approx 50 - 100$. Recalling the definition of the skin friction coefficient, $c_f = \tau_w/(\rho U^2/2)$, and using the relation given in Eq.(8), an expression in terms of Re_x can be obtained

$$\left(\frac{\Delta x}{\Delta y}\right)_{overlap} \approx \frac{1}{\sqrt{2}} \frac{n_y^w}{n_x} \frac{b}{l_y^+} \sqrt{\frac{7b}{36}(\alpha+1)} Re_x^{1+3\alpha/2}. \quad (15)$$

Therefore, for any value of $\alpha > -2/3$ the mesh anisotropy, $\Delta x/\Delta y$, tends to grow with Re_x. Taking typical values for $n_x = 10$, $n_y^w = 20$ and $l_y^+ = 100$, and using, respectively, the skin friction coefficient correlations used by Chapman [2], *i.e.* $\alpha = -1/5$ and $b = 0.37$, and Choi and Moin [3], *i.e.* $\alpha = -1/7$ and $b = 0.17$, it simplifies

$$\left(\frac{\Delta x}{\Delta y}\right)_{overlap}^{Chapman} \approx 0.00125 Re_x^{7/10}\ ; \quad \left(\frac{\Delta x}{\Delta y}\right)_{overlap}^{Choi\&Moin} \approx 4.047 \times 10^{-4} Re_x^{11/14}. \quad (16)$$

Just as examples, this leads to mesh anisotropies of 19.9 and 20.96 at $Re_x = 10^6$, and 99.7 and 127.97 at $Re_x = 10^7$. Therefore, numerical techniques that behave robustly in such meshes are of great interest. In this context, a new definition of the subgrid characteristic length in presented and tested in the next section.

3 A new definition of the subgrid characteristic length

Because of its inherent simplicity and robustness, the eddy-viscosity assumption, $\tau(\overline{u}) \approx -2\nu_e S(\overline{u})$, is by far the most used closure model for LES equations (1). Then, the eddy-viscosity, ν_e, is usually modeled as follows

$$\nu_e = (C_m \Delta)^2 D_m(\overline{u}). \quad (17)$$

In the last decades most of the research has focused on either the calculation of the model constant, C_m (*e.g.* the dynamic modeling approach), or the development of more appropriate model operators $D_m(\overline{u})$ (*e.g.* WALE [4], Vreman's [5], Verstappen's [6], σ-model [7], S3PQR [1],...). Surprisingly, little attention has been paid on the computation of the subgrid characteristic length, Δ, which is also a key element of any eddy-viscosity model. Despite the fact that in some situations it may provide very inaccurate results, three and a half decades later, the approach proposed by Deardorff [8], *i.e.*, the cube root of the cell volume (see Eq. 20), is by far the most widely used to computed the subgrid characteristic length, Δ. Its inherent simplicity and applicability to unstructured meshes is probably a very good explanation for that. With the aim to overcome the limitations of the Deardorff definition, the following definition is proposed

$$\Delta_{\mathrm{lsq}} = \sqrt{\frac{\mathsf{G}_\Delta \mathsf{G}_\Delta^T : \mathsf{G}\mathsf{G}^T}{\mathsf{G}\mathsf{G}^T : \mathsf{G}\mathsf{G}^T}}, \quad (18)$$

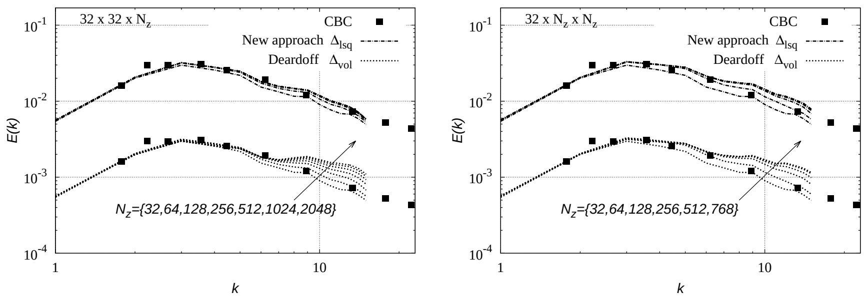

Fig. 1 Energy spectra for decaying isotropic turbulence corresponding to the experiment of Comte-Bellot and Corrsin [9]. Results obtained with the new definition Δ_{lsq} proposed in Eq.(18) are compared with the classical definition proposed by Deardorff given in Eq.(20). For clarity, latter results are shifted one decade down.

where $\mathsf{G} \equiv \nabla \overline{u}$, $\mathsf{G}_\Delta \equiv \mathsf{G}\Delta$ and $\Delta \equiv \mathrm{diag}(\Delta x, \Delta y, \Delta z)$ (for a Cartesian grid). This definition of Δ fulfills a set of desirable properties. Namely, it is locally defined and well bounded, $\Delta x \leq \Delta_{\mathrm{lsq}} \leq \Delta z$ (assuming that $\Delta x \leq \Delta y \leq \Delta z$). Moreover, it is sensitive to flow orientation and applicable to unstructured meshes (by simply replacing the tensor Δ by the Jacobian of the mapping from the physical to the computational space). This definition (18) is obtained minimizing (in a least-squares sense) the difference between the leading terms of the Taylor series of the SGS tensor, $\tau(\overline{u})$, for an isotropic and an anisotropic filters lengths; namely,

$$\tau(\overline{u}) = \frac{\Delta^2}{12}\mathsf{G}\mathsf{G}^T + \mathscr{O}(\Delta^4) \; ; \quad \tau(\overline{u}) = \frac{1}{12}\mathsf{G}_\Delta \mathsf{G}_\Delta^T + \mathscr{O}(\Delta^4), \tag{19}$$

Results displayed in Figure 1 correspond to the classical experimental results obtained by Comte-Bellot and Corrsin [9]. LES results have been obtained using the Smagorinsky model, for a set of (artificially) stretched meshes. Regarding the spatial discretization of the eddy-viscosity models, the approach proposed in Ref. [10] has been used. Results for pancake-like meshes with $32 \times 32 \times N_z$ and $N_z = \{32, 64, 128, 256, 512, 1024, 2048\}$ are displayed in Figure 1 (top): for increasing values of N_z the results obtained using the Deardorff definition, given in

$$\Delta_{\mathrm{vol}} = (\Delta x \Delta y \Delta z)^{1/3}, \tag{20}$$

diverge. This is because the value of Δ tends to vanish and, therefore, the subgrid-scale models switch off. This is not the case for the definition of Δ proposed in this work. Instead, results rapidly converge for increasing values of N_z. A similar behavior is observed in Figure 1 (bottom) for pencil-like meshes with $32 \times N_z \times N_z$ and $N_z = \{32, 64, 128, 256, 512, 768\}$. Therefore, the proposed definition of the subgrid characteristic length, Δ_{lsq}, seems to minimize the effect of mesh anisotropies on the performance of subgrid-scale models.

4 Concluding remarks

Estimations of the computational costs for LES with and without wall modeling were originally given by Chapman [2], and more recently, by Choi and Moin [3]. Here, these estimations have been extended by the general power-law of the skin friction coefficient given in Eq.(7), including the temporal scales. Furthermore, it has been found that the mesh anisotropy in the overlap region increases with the Reynolds number (see Eq.15). This represents an additional challenge for LES. In this context, a novel definition of subgrid characteristic length, Δ, is proposed with the aim to answer the following research question: *can we find a simple and robust definition of Δ that minimizes the effect of mesh anisotropies on the performance of SGS models?* In this regard, we consider the novel definition of Δ_{lsq} proposed in Eq.(18) as a very good candidate. Results for decaying isotropic turbulence show that the proposed definition of Δ seems to minimize the effect of mesh anisotropies on the performance of subgrid scale models.

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