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ON A CONSERVATIVE SOLUTION TO CHECKERBOARDING: EXAMINING THE DISCRETE LAPLACIAN KERNEL USING MESH CONNECTIVITY

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INTRODUCTION

CFD codes that are used for industrial applications commonly use a collocated grid arrangement to calculate the physical flow variables. The main advantages of this arrangement, in comparison to the staggered one, are the possibility to extend the solution domain to more complex geometries and a more efficient data structure, which are both of great importance in industry. When using a central differencing scheme to discretise the continuous operators of the Navier-Stokes equations, a wide stencil is obtained for the Laplacian operator. This wide stencil, in turn, leads to a decoupling between odd and even grid points of the pressure field that results from the pressure Poisson equation. This decoupling can lead to nonphysical, spurious modes in the solution, a problem commonly known as the checkerboard problem [1].

Generally, this problem is avoided by using a compact stencil Laplacian. A method to do so was first developed by Rhie and Chow [2]. This method solves the problem of decoupled grid points and eliminates the possibility of spurious modes in the pressure field. However, this method introduces nonphysical, numerical dissipation of kinetic energy [2, 3]. This dissipation disrupts the delicate interaction between convective transport and physical dissipation, especially at the smallest scales of motion. By doing so, it becomes impossible to capture the essence of turbulence, which is of high importance in accurate LES and DNS simulations [4].

Many commercial codes favour the extra stability that this method offers at the price of a lower accuracy. Unconditional stability, however, can also be achieved by mimicking the underlying symmetry properties of the continuous operators of the Navier-Stokes equations, when discretising them. A method that does this was developed for staggered Cartesian grid arrangements by Verstappen and Veldman [4] and later extended to collocated unstructured grids by Trias *et al.* [5]. Since the kinetic energy is conserved and stability is unconditional, using the method of Rhie and Chow comes at a higher price and an alternative method should be sought after. One method mentioned here is the one described by Larsson and Iaccarino [6], in which the kernel of the discrete Laplacian operator matrix is determined and used to eliminate the spurious modes. However, on non-Cartesian grids, this method involves performing a singular value decomposition (SVD), for which the computational cost grows exponentially with the number of grid points, as $\mathcal{O}(N_{grid}^3)$, making this method nonviable for industrial applications [7].

In this work a conservative solution to the checkerboard problem will be examined by looking at the relation between the connectivity of the mesh and the kernel of the discrete Laplacian operator matrix. By understanding this relation better, a prediction can be made for a set of vectors that span the nullspace, and by projecting the pressure solution field onto this nullspace, the spurious modes can be eliminated. The relation between the mesh and the kernel will first be examined, after which a method is developed to predict and remove spurious modes, this method is then tested and compared to the conventional Rhie-Chow interpolation method. The new method could prove to be especially useful in the field of magnetohydrodynamics, where a second Poisson equation for the electric potential has to be solved and stability and conservative properties are important to accurately balance high opposing forces.

RELATION BETWEEN THE MESH AND THE KERNEL

Let the discrete wide stencil Laplacian operator be denoted by $L_c = M\Gamma_{cs}\Gamma_{sc}G$, which follows the discretisation of Trias *et al.* [5] and is a chain operations: (1) face gradient, (2) face-to-cell interpolation, (3) cell-to-face interpolation and (4) divergence. The first thing to note is that the gradient at cell i is given by:

$$\begin{aligned} \left[\Gamma_{sc}G\phi_c\right]_i &= \frac{1}{2\left[\Omega_c\right]_{i,i}} \sum_{f\in F_f(i)} \left[\Omega_s\right]_{f,f} \frac{\phi_n-\phi_i}{\delta_{nf}}\mathbf{n}_{if} \\ &= \frac{1}{2\left[\Omega_c\right]_{i,i}} \sum_{f\in F_f(i)} A_f\phi_n\mathbf{n}_{if} \end{aligned} \tag{1}$$

because the sum of outward pointing face area vectors always equals 0:

$$-\frac{1}{2\left[\Omega_c\right]_{i,i}}\phi_i \sum_{f\in F_f(i)} A_f\mathbf{n}_{if} = 0 \tag{2}$$

Therefore, the value in the central cell i does not contribute to the gradient in the central cell itself. Similarly, the cell-centered divergence at cell i is given by:

$$\begin{aligned} \left[M\Gamma_{cs}\boldsymbol{\psi}_c\right]_i &= \frac{1}{2} \sum_{f\in F_f(i)} (\boldsymbol{\psi}_i+\boldsymbol{\psi}_n)\cdot\mathbf{n}_{if}A_f \\ &= \frac{1}{2} \sum_{f\in F_f(i)} \boldsymbol{\psi}_n\cdot\mathbf{n}_{if}A_f \end{aligned} \tag{3}$$

Again, the value in the central cell i does not contribute to the divergence in the central cell itself. L_c is a sequence of both operators and will therefore only potentially connect cell i to cell k, if they share a neighbour j:

$$[L_c]_{i,k} = \sum_j \frac{1}{4\,[\Omega_c]_j} \left(A_{i,j}\mathbf{n}_{i,j}\right) \cdot \left(A_{j,k}\mathbf{n}_{j,k}\right) \tag{4}$$

If we use $A_{j,i}\mathbf{n}_{j,i} = -\sum_{k\neq i} A_{j,k}\mathbf{n}_{j,k}$, then we can verify that L_c should be negative-definite symmetrical, with columns and rows summing to 0:

$$\begin{aligned}[L_c]_{i,i} &= \sum_j \frac{1}{4\,[\Omega_c]_j} \left(A_{i,j}\mathbf{n}_{i,j}\right) \cdot \left(A_{j,i}\mathbf{n}_{j,i}\right) \\ &= -\sum_{k\neq i}\sum_j \frac{1}{4\,[\Omega_c]_j} \left(A_{i,j}\mathbf{n}_{i,j}\right) \cdot \left(A_{j,k}\mathbf{n}_{j,k}\right)\end{aligned} \tag{5}$$

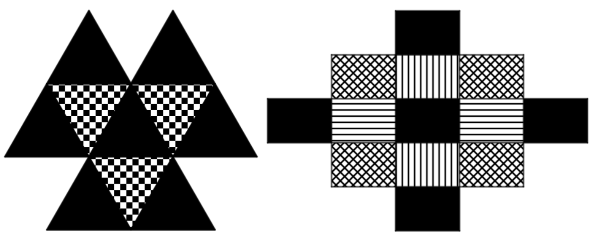
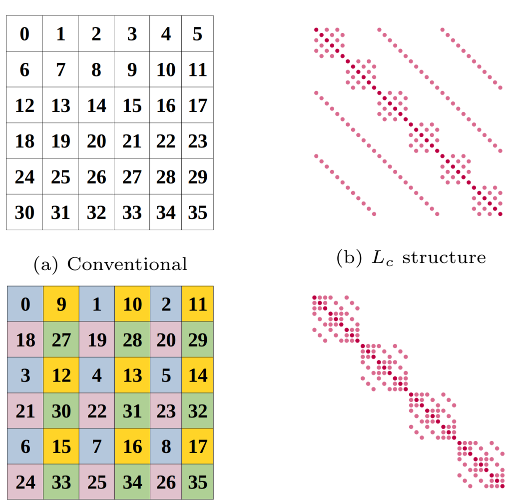


Figure 1: 2 disconnected cell groups in a regular triangular mesh (left) and $2^{dim} = 4$ in a Cartesian mesh (right)

Now we can predict from equation (4) that an extra vector, in addition to the constant vector, is needed to span the nullspace of L_c, if an odd-even parity can be established in the mesh, as seen in figure 1. In the special case that the dot product in (4) equals zero due to orthogonal faces, there will not be any connection. Therefore, in Cartesian meshes, no diagonal connections exist and the number of disconnected cell groups will equal 2^{dim}, as will the rank of the nullspace of L_c.



(a) Conventional

(b) L_c structure

(c) Deliberate

(d) Block diagonalised

Figure 2: Representation of Laplacian matrix entries for a Cartesian 6×6 mesh, with cyclic boundaries

The disconnected groups can also be visualised with a block diagonalisation of the Laplacian matrix. When using conventional cell numbering, as seen in figure 2a, the disconnection is not immediately apparent from the matrix in figure 2b. When renumbering the cells deliberately, as done in figure 2c, the separation is evident from the block diagonal matrix, figure 2d. This block diagonalisation is not possible for the compact Laplacian. One final way to express this relation, in terms of graph connectivity, is that the rank of the nullspace equals the number of connected components in the graph Laplacian, which is constructed using mesh connectivity. Since the connectivity can be derived from the mesh and is tied to the kernel of L_c, the spurious modes can be removed from the solution while avoiding the high cost of the SVD.

COMPUTATIONAL VERIFICATION

A solver is developed with symmetry-preserving discretisation of the continuous operators [5], with optionally compact or wide stencil Laplacian in the pressure Poisson equation. A priori a set of nullspace spanning vectors will be predicted from the mesh connectivity, so that spurious modes can be removed in calculating the pressure field. A Taylor-Green vortex and a turbulent channel flow, figure 3, will be used to verify the method, by monitoring evolution of kinetic energy and the presence of spurious modes. It is expected that the compact stencil Laplacian will show numerical dissipation and that the wide stencil Laplacian will show checkerboarding, whereas the new approach solves both problems.

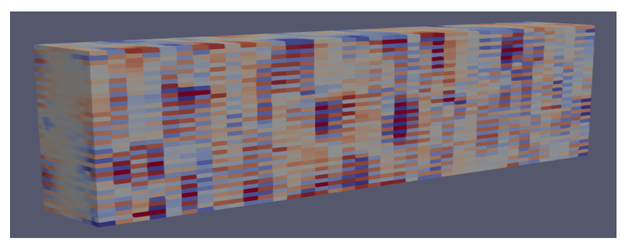


Figure 3: Channel flow with checkerboarding

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