A conservative solution to checkerboarding:
Examining the discrete Laplacian kernel using mesh connectivity

J.A. Hopman, F.X. Trias and J.Rigola

Heat and Mass Transfer Technological Center (CTTC)

Technical University of Catalonia (UPC), Terrassa, Spain

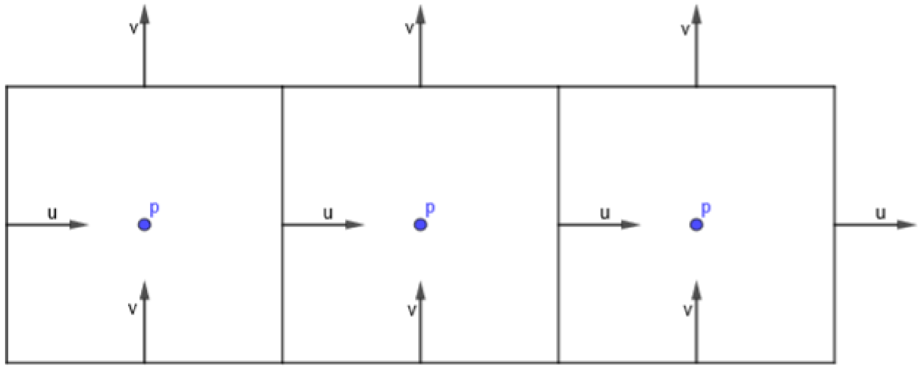
DLES13 2022 – Direct and Large Eddy Simulation
26-28 October 2022, Udine, Italy

Pressure-velocity coupling

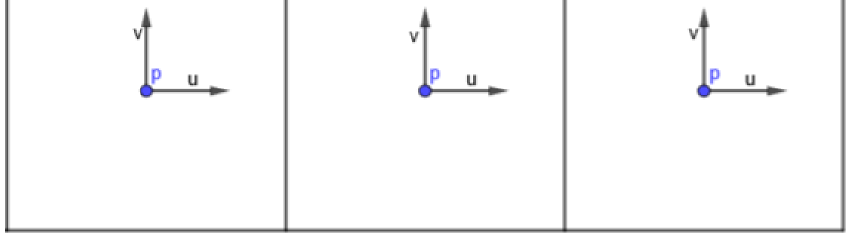
$$\mathbf{u}^{n+1} = \mathbf{u}^{p} - \nabla \mathbf{p}^{n+1}$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0$$

$$\nabla^2 \mathbf{p}^{n+1} = \nabla \cdot \mathbf{u}^{p}$$



staggered



collocated

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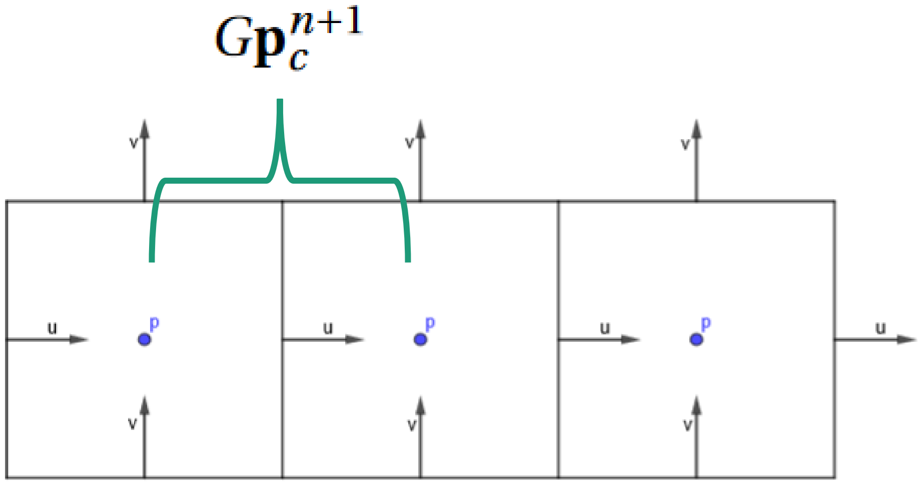

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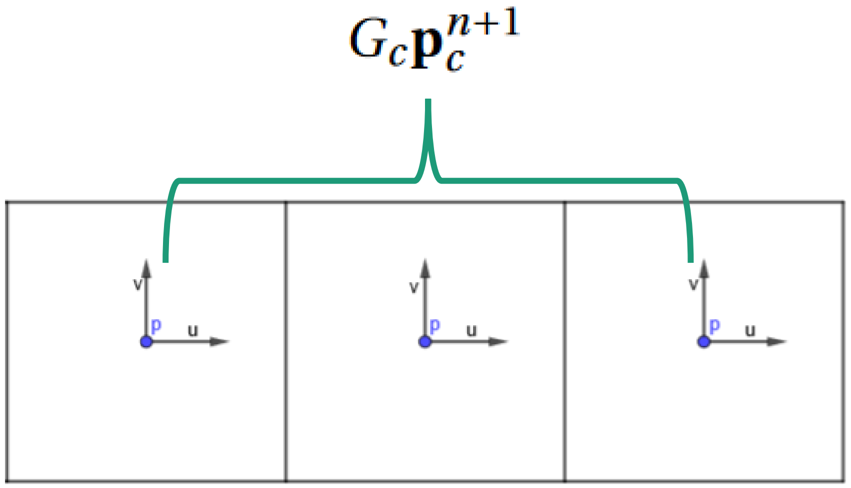

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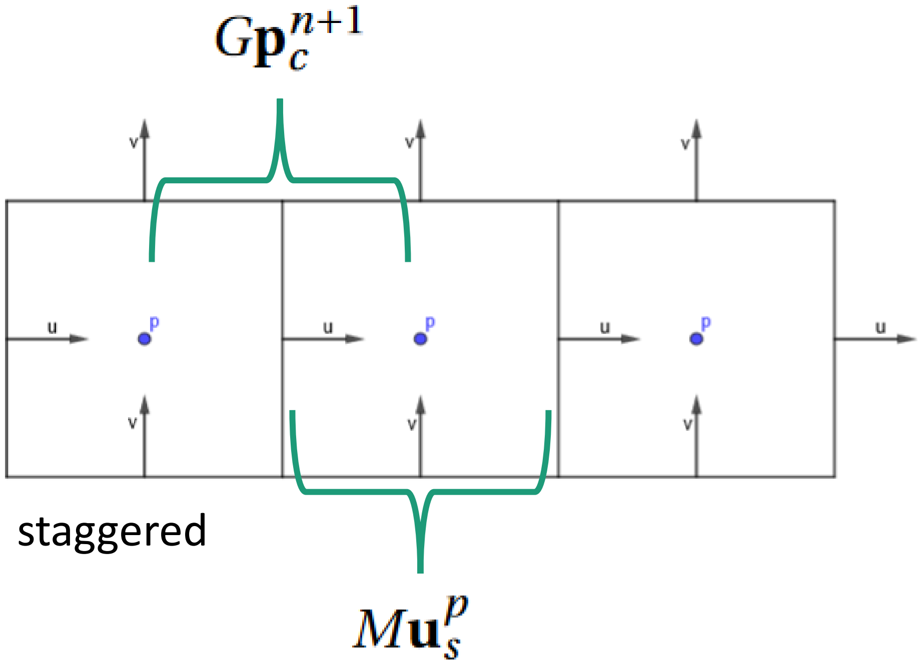

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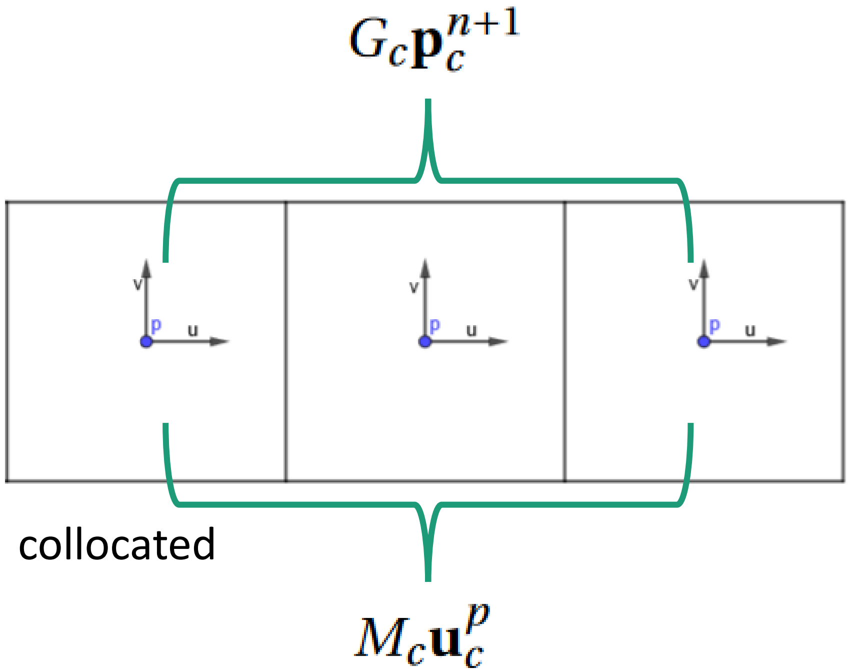

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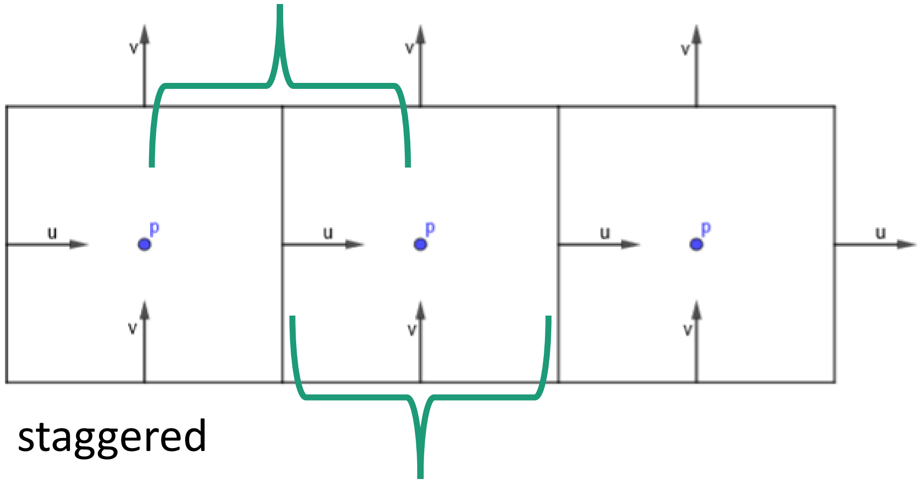
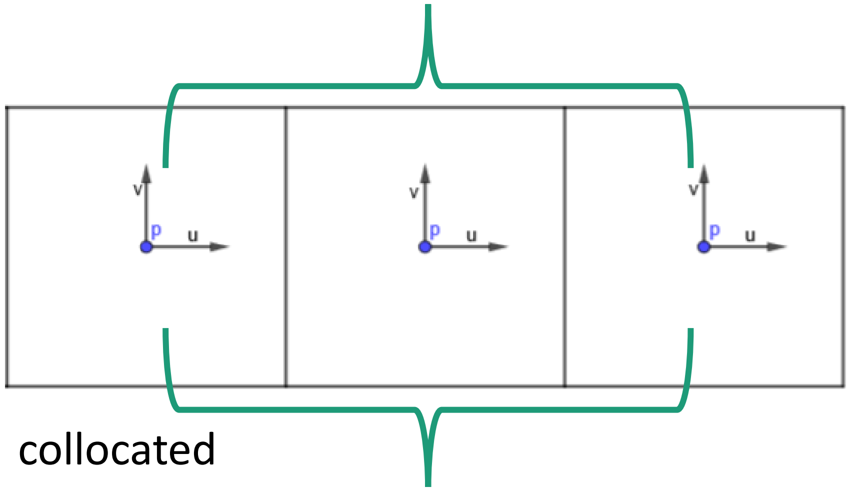
$G\mathbf{p}_c^{n+1}$



$M\mathbf{u}_s^{p}$

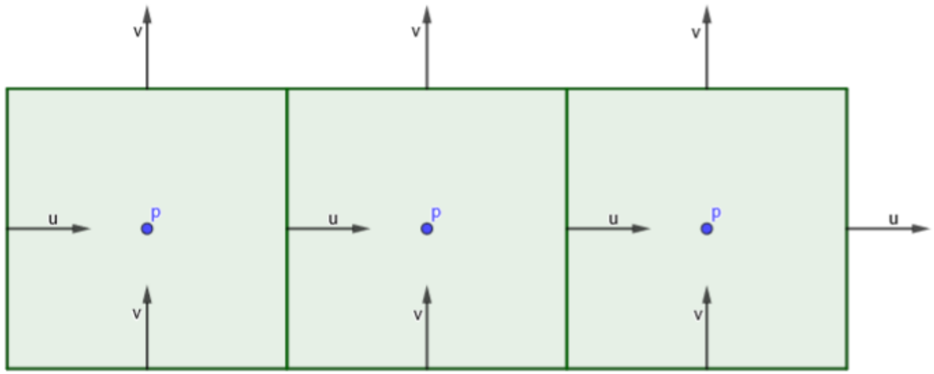
	Stag	Col
Grad	G	G_c
Div	M	M_c

$G_c\mathbf{p}_c^{n+1}$



$M_c\mathbf{u}_c^{p}$

Compact vs wide stencil Laplacian



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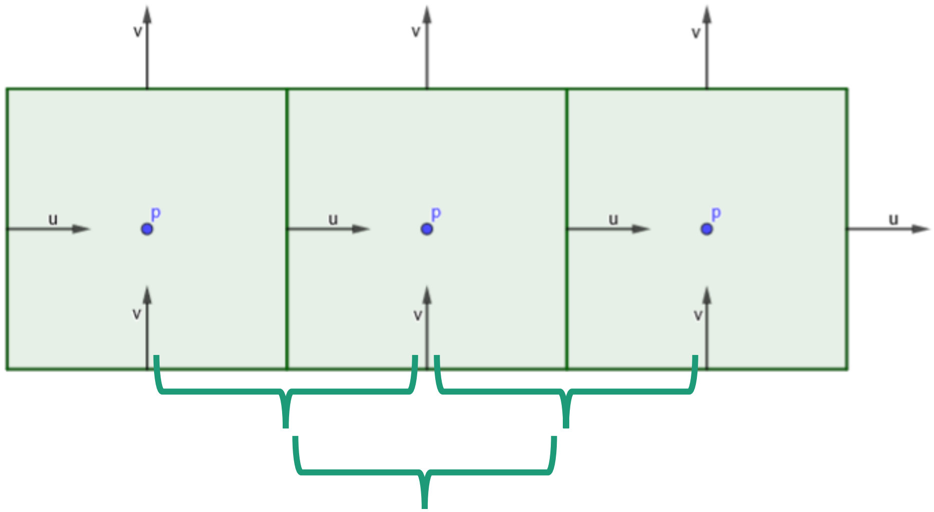
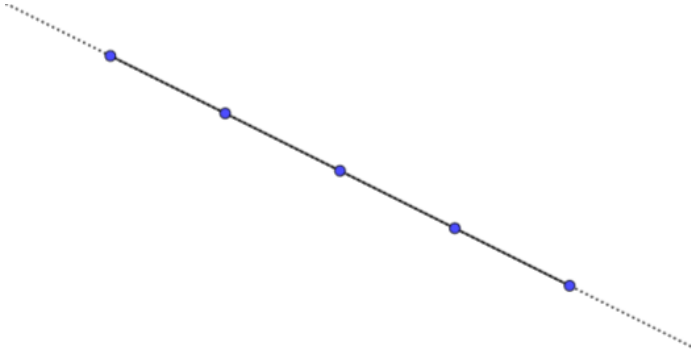


$L = MG$ compact stencil



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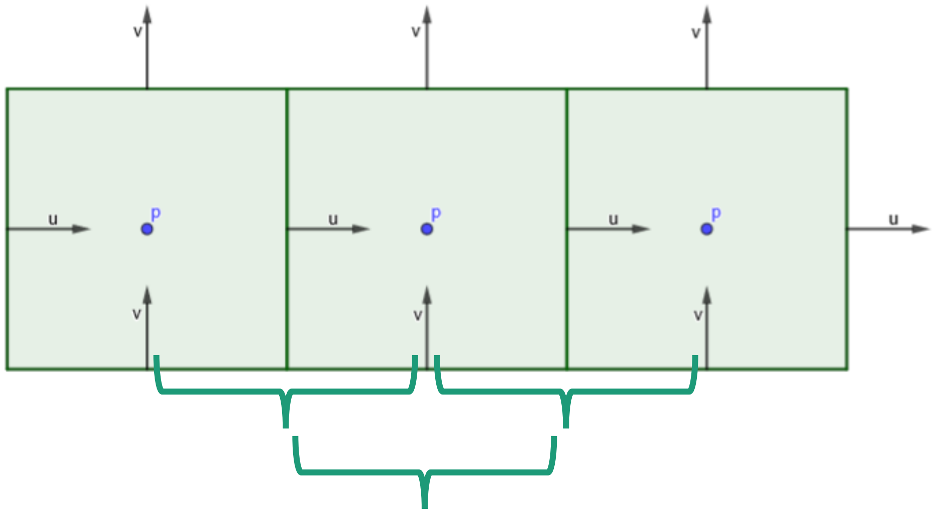

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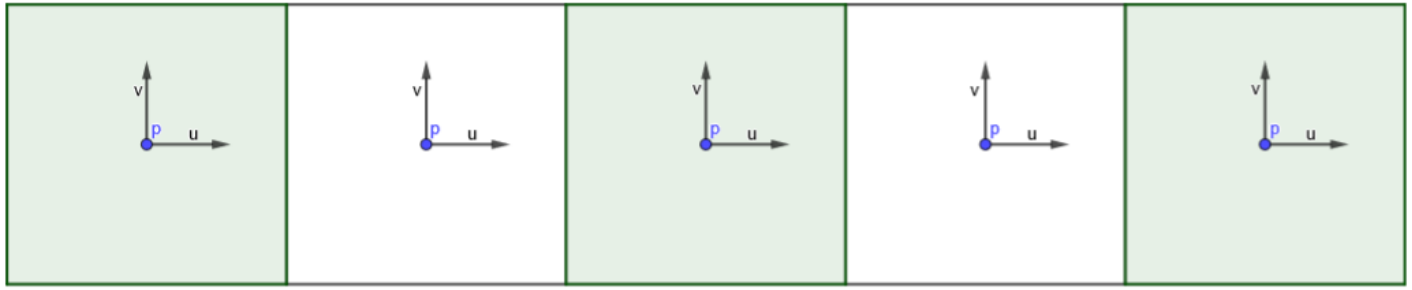

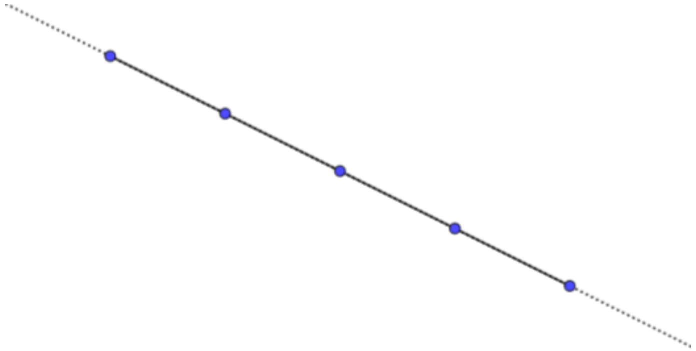

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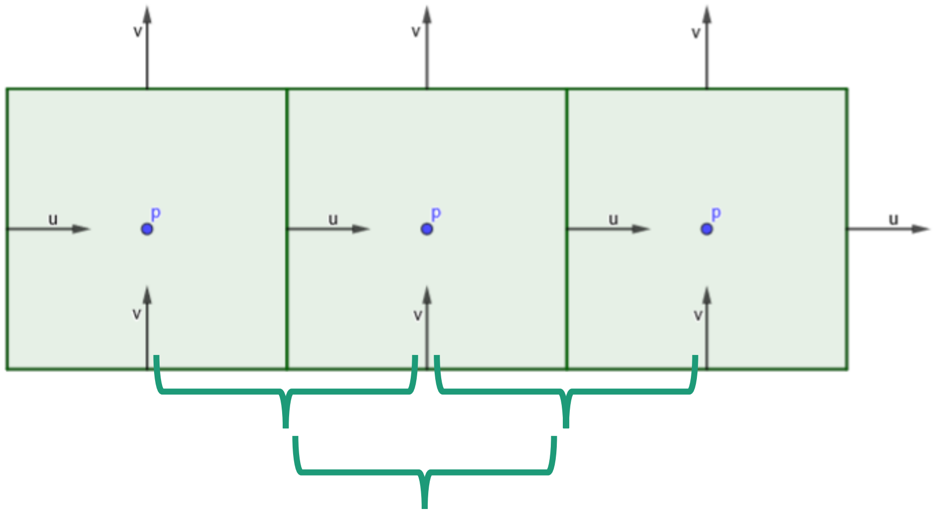

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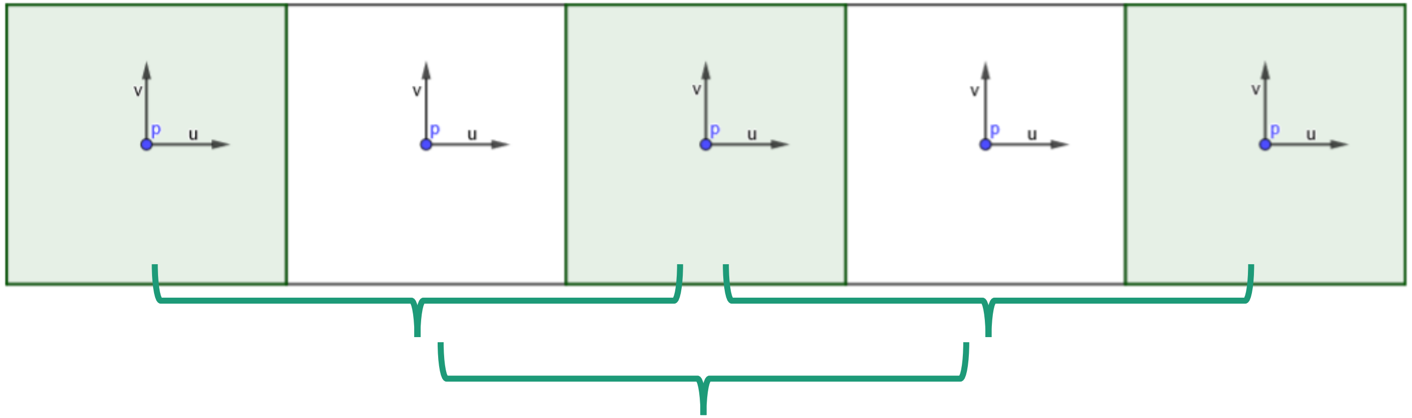
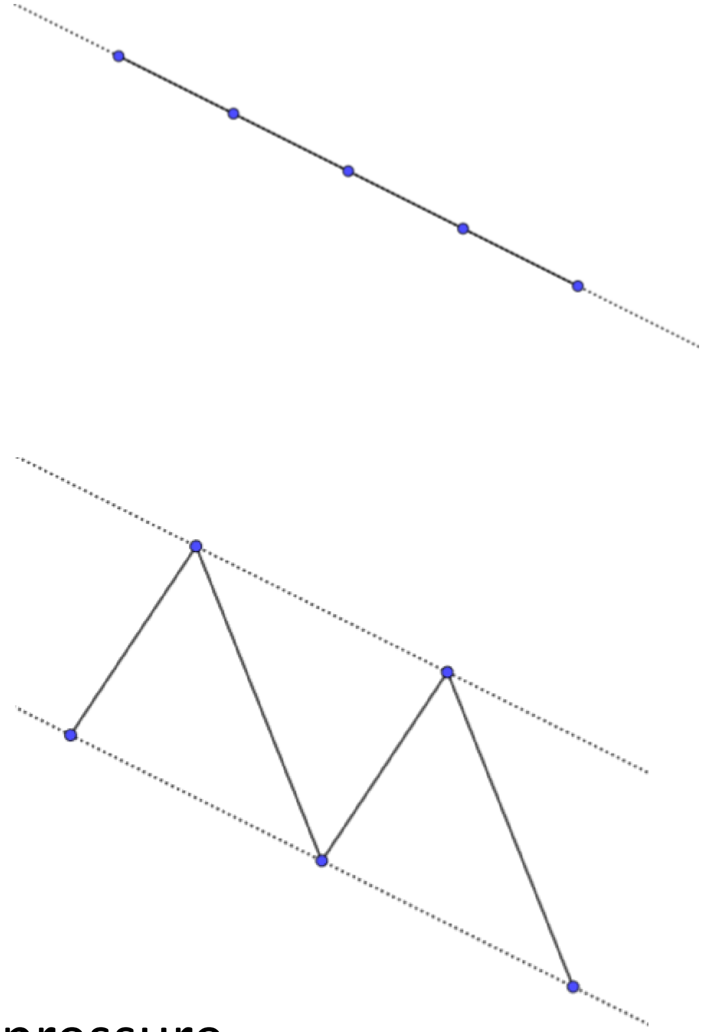


$L_c = M_c G_c$ wide stencil → decoupling of pressure



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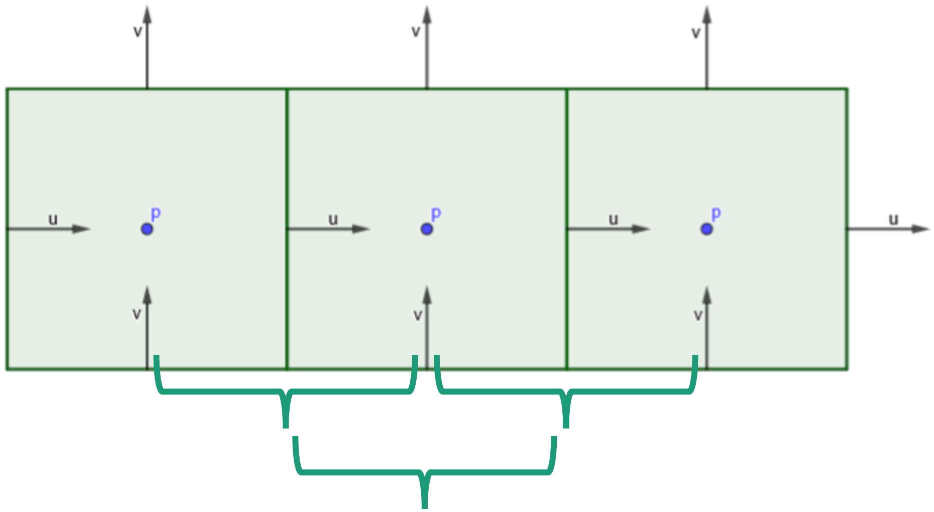

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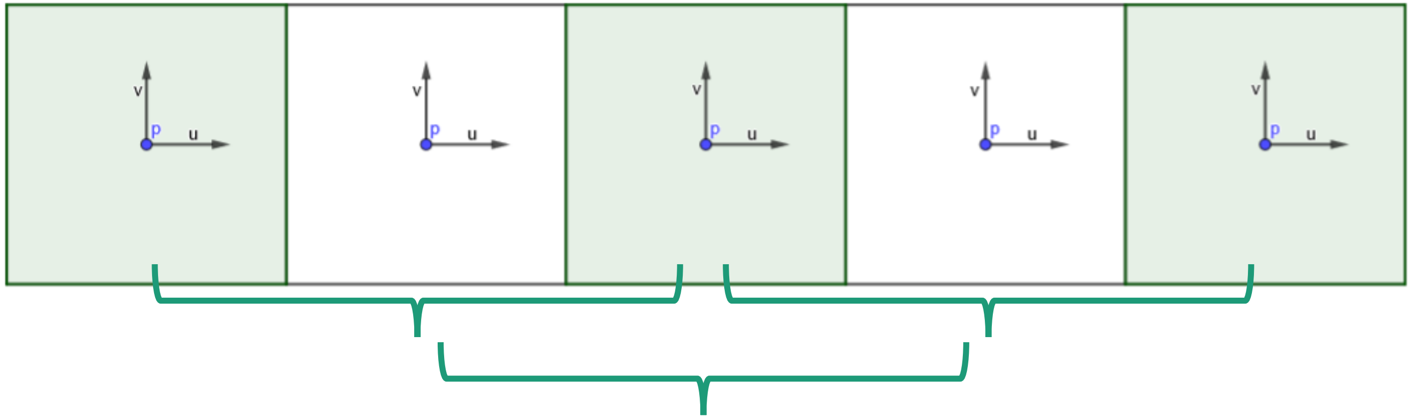

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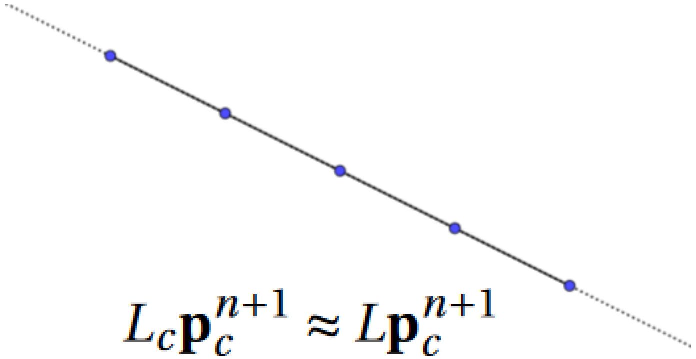
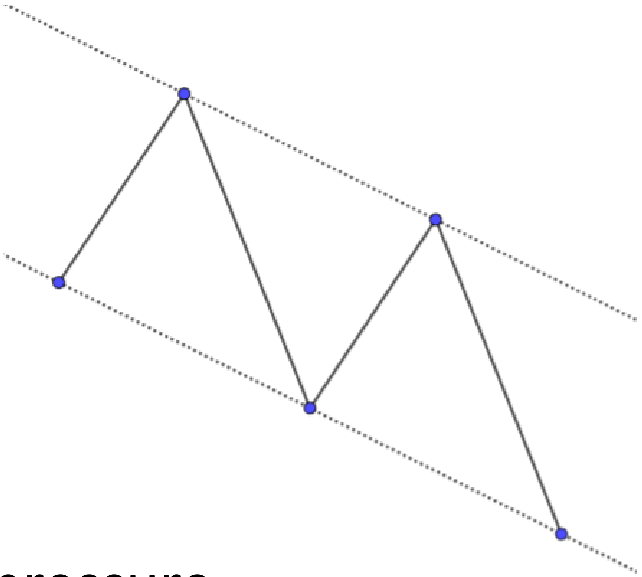


$L_c \mathbf{p}_c^{n+1} \approx L \mathbf{p}_c^{n+1}$

→ pressure error



Motivation

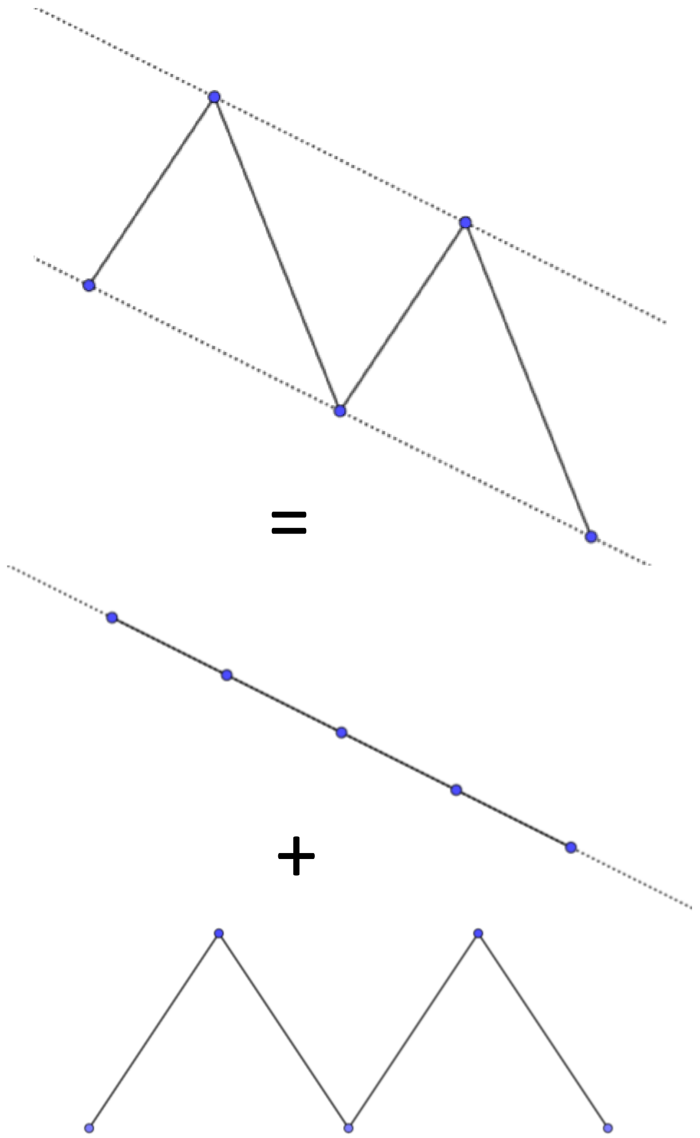
Can we remove spurious modes without pressure error?

Develop proper filtering for Cartesian meshes

Gain insight in filtering for unstructured meshes

Spurious modes – Kernel of Laplacian

$$\mathbf{p}_c = \mathbf{p}_c^+ + \mathbf{p}_c^-$$
$$\mathbf{p}_c^- \in \mathrm{Ker}(L_c)$$
$$L_c\mathbf{p}_c = L_c\mathbf{p}_c^+ + L_c\mathbf{p}_c^- = L_c\mathbf{p}_c^+$$



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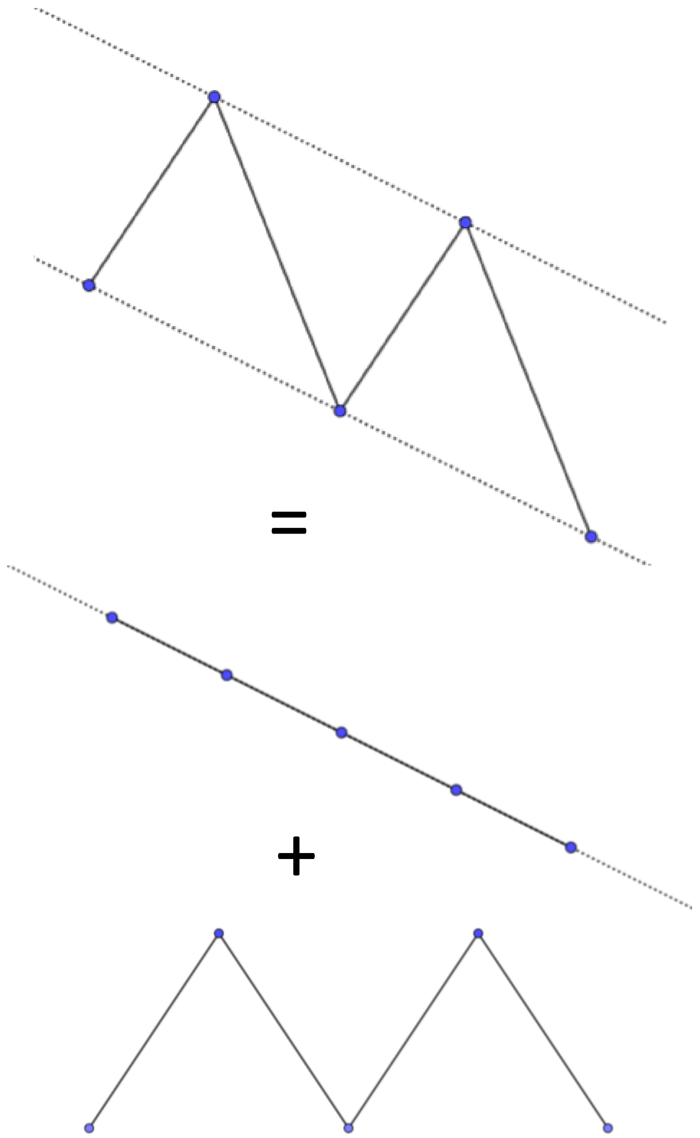
Nullity(L_c) > 1→ spurious modes
(Nullity(L) = 1: constant mode → reference pressure)

If we know Ker(L_c) we can just filter $\mathbf{p}^-$



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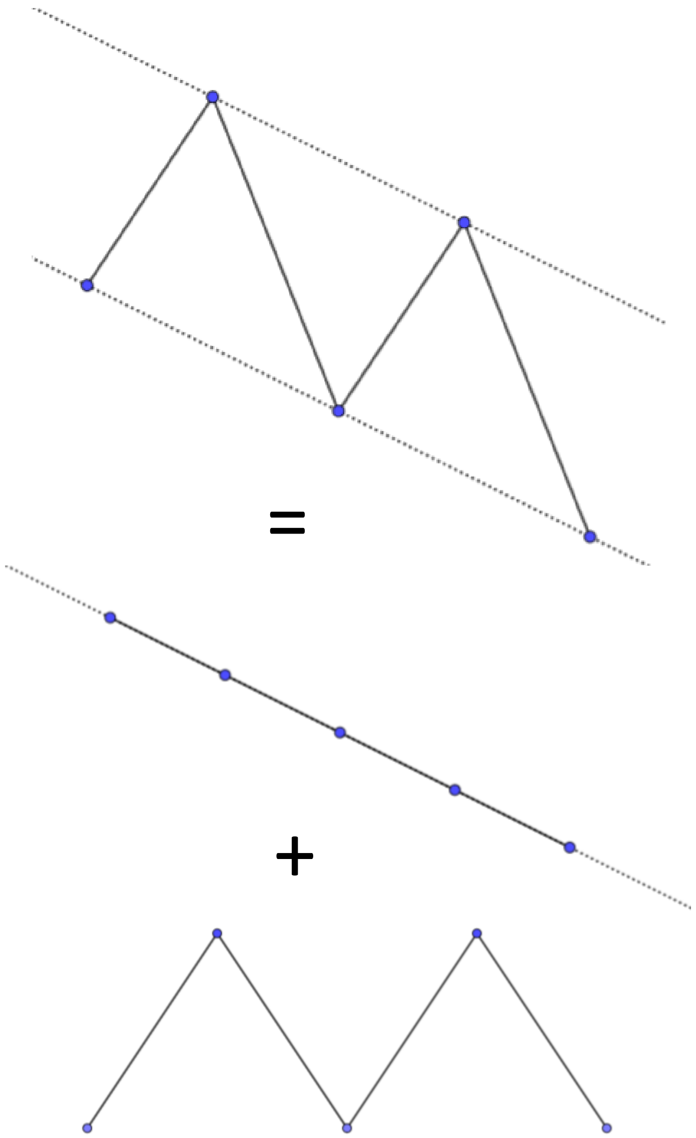

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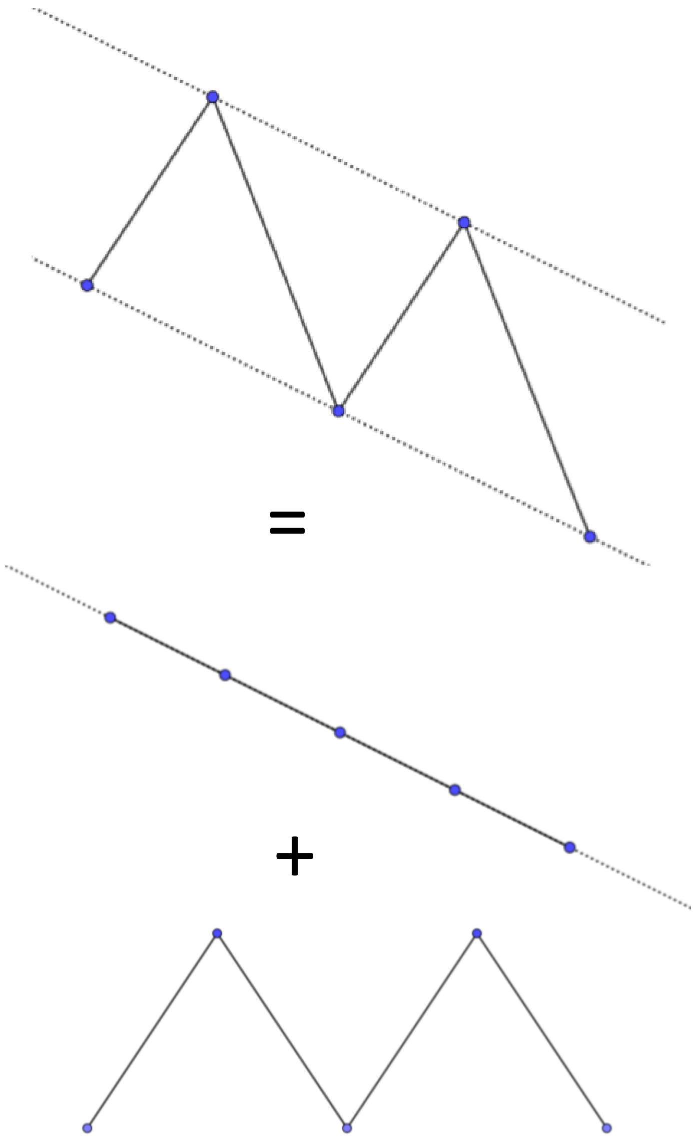

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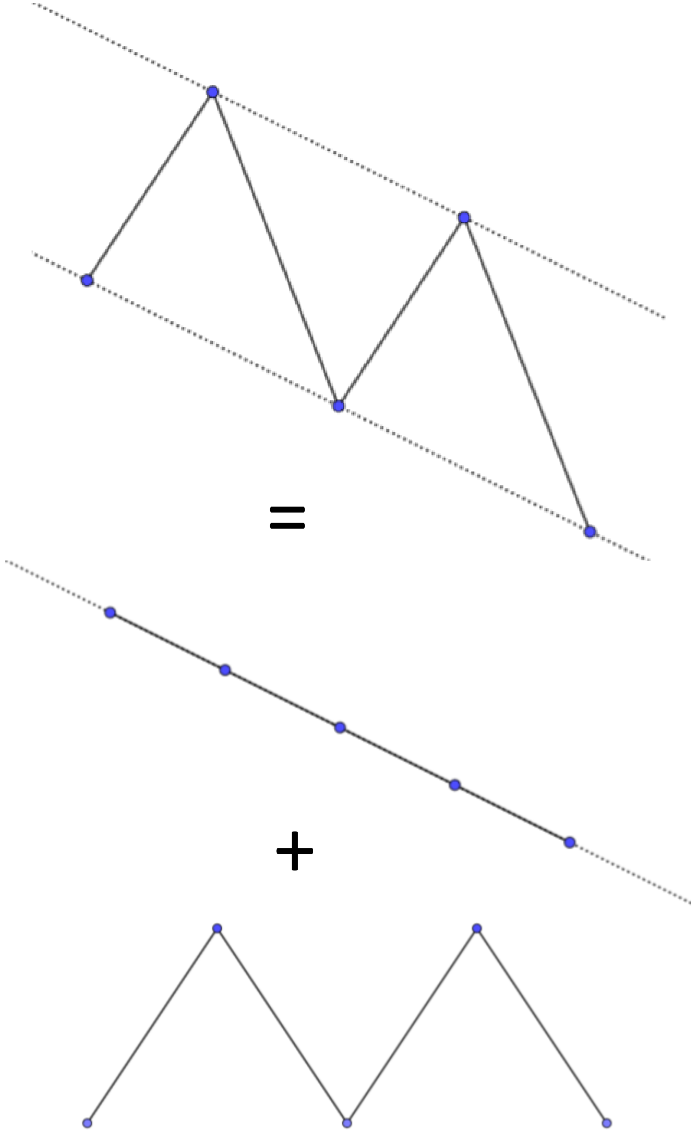

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L_c depends on mesh and discretisation;
can't we deduce Ker(L_c) from mesh and discretisation?

Laplacian matrix – interpolators

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<u>Interpolators related, 1 d.o.f.</u>

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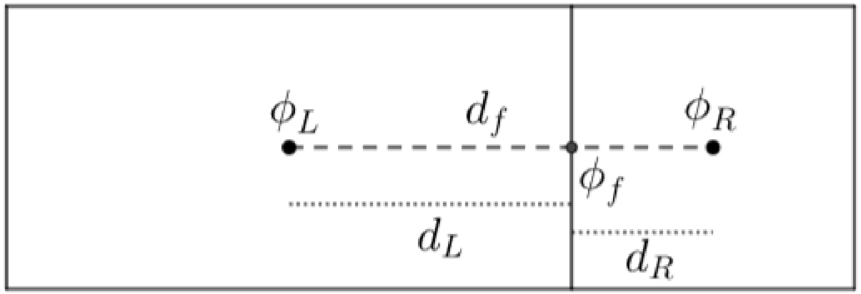

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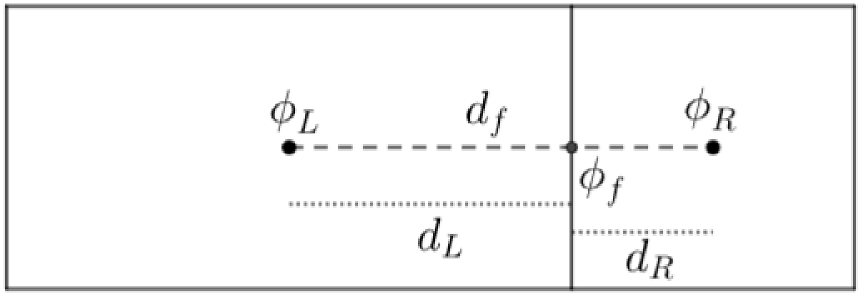

Midpoint: $\Pi_{cs}^M \to \phi_f = \dfrac{\phi_L + \phi_R}{2}$

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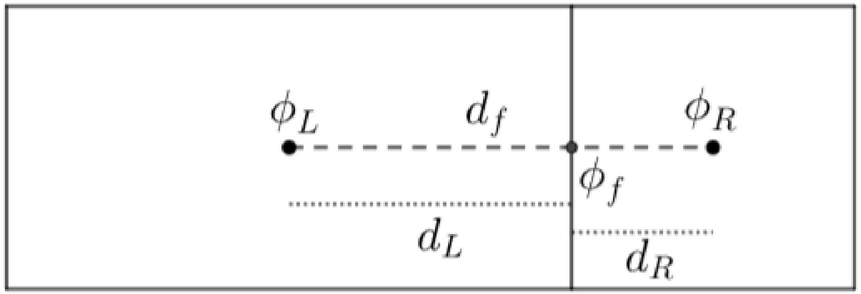

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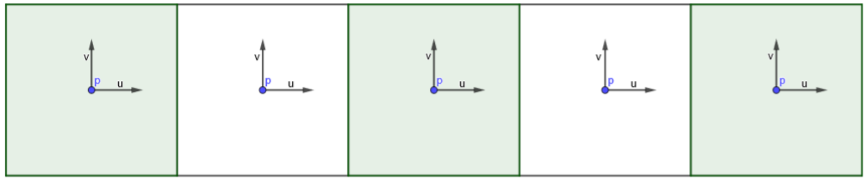

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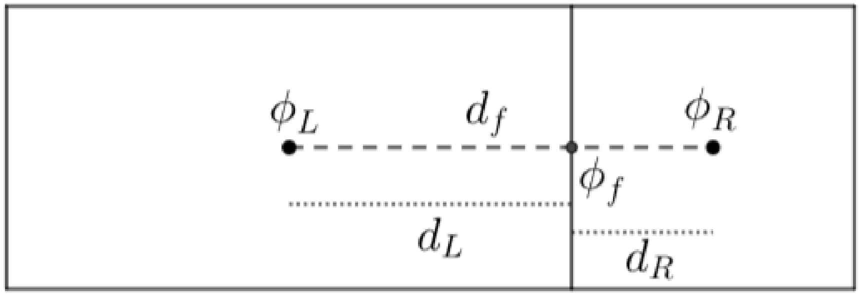

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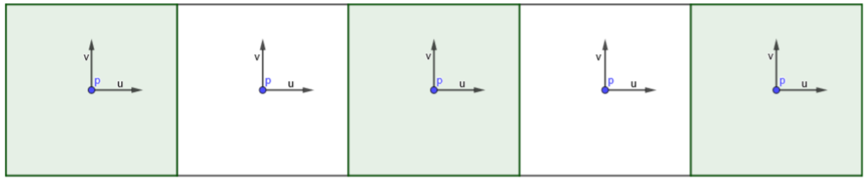

Interpolators related, 1 d.o.f.

Midpoint results in checkerboard

Relation between Ker(L_c) and mesh – Midpoint

$$[L_c]_{i,k} = \sum_j \frac{1}{4\,[\Omega_c]_j} \left(A_{i,j}\mathbf{n}_{i,j}\right) \cdot \left(A_{j,k}\mathbf{n}_{j,k}\right)$$

Group together the second neighbours

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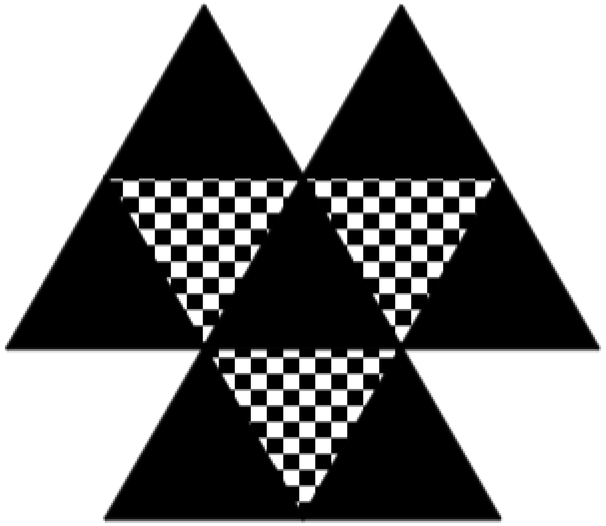

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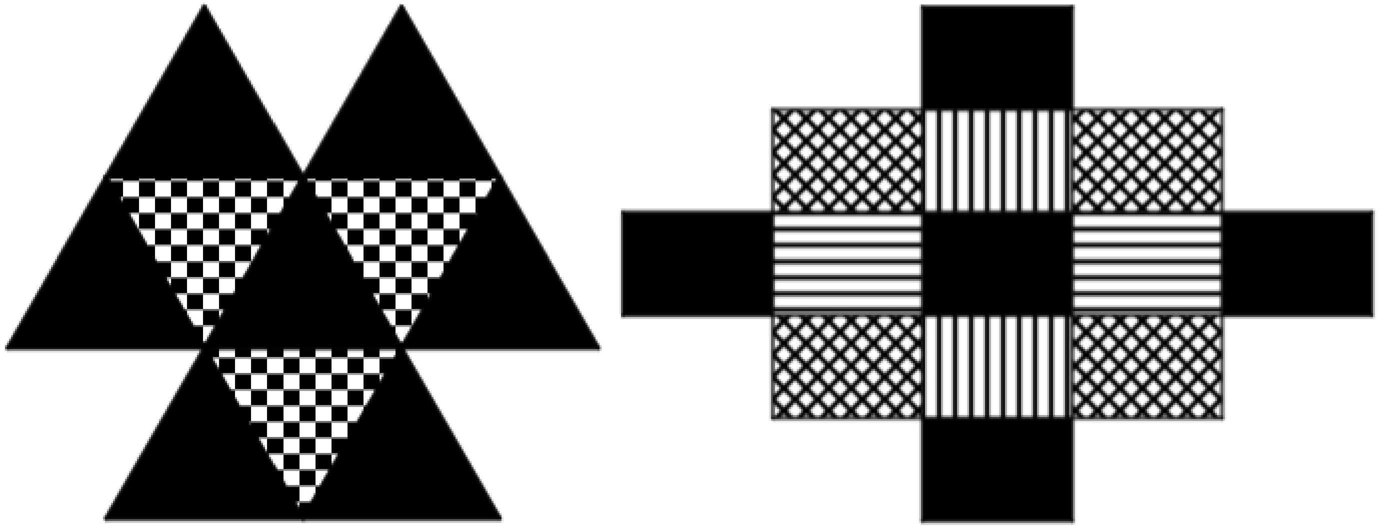

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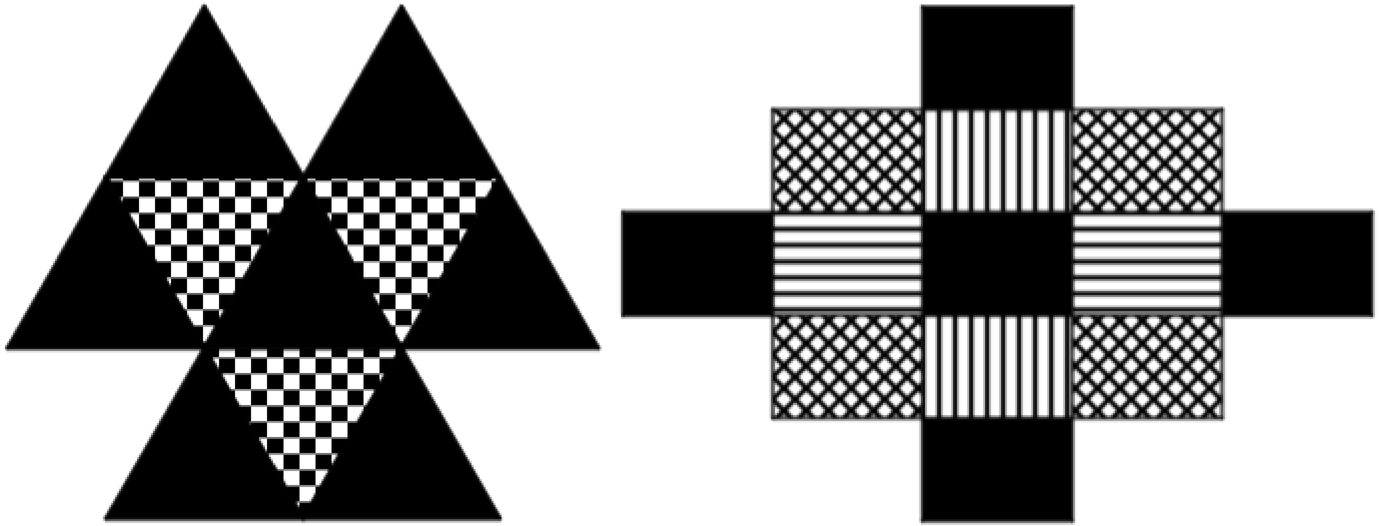
Group together the non-orthogonal second neighbours

= 0 for orthogonal faces → Cartesian meshes



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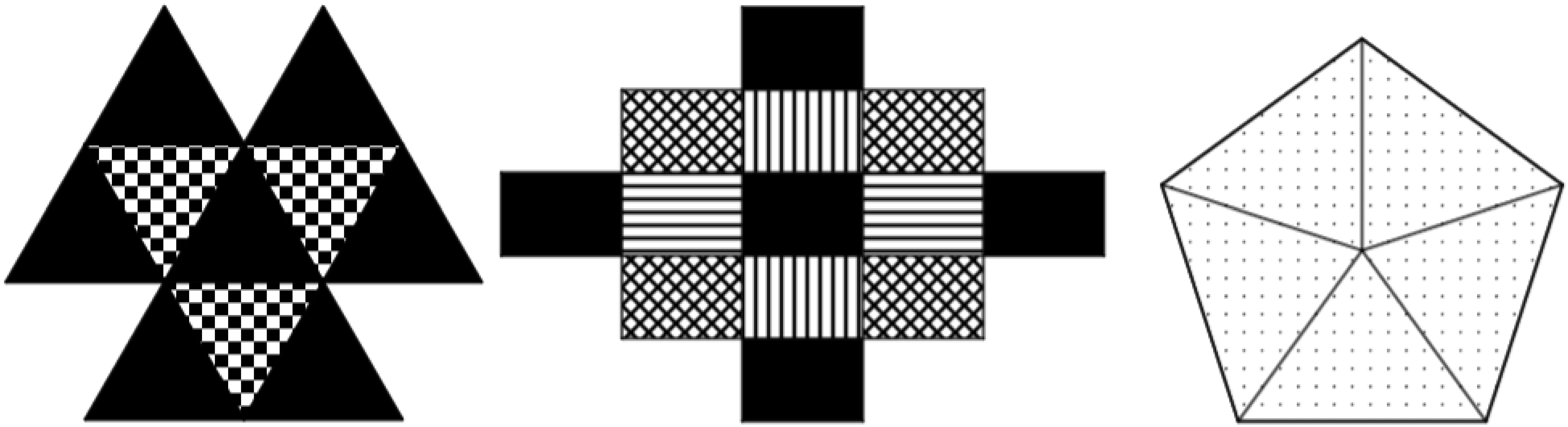

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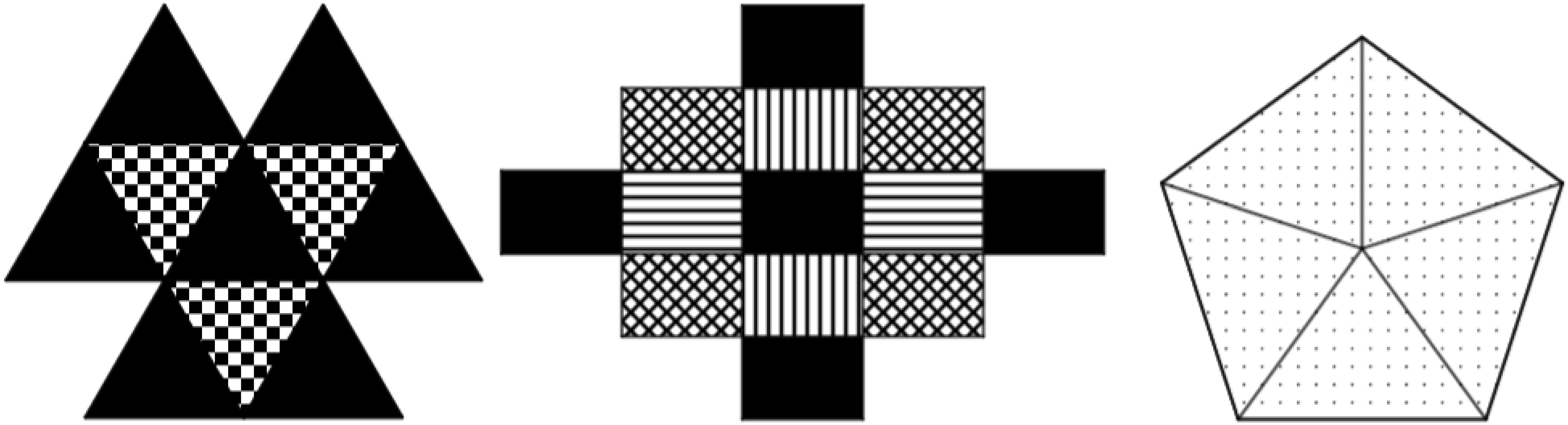

Vectors that span the kernel can be derived from the mesh.
Nullity = number of disconnected cell groups

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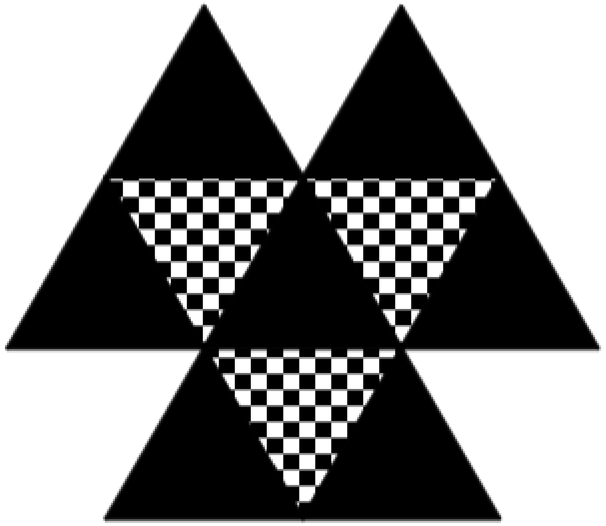
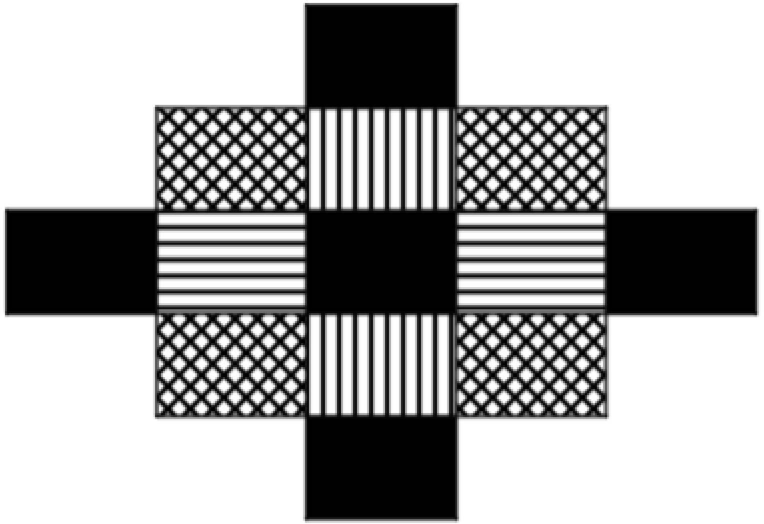
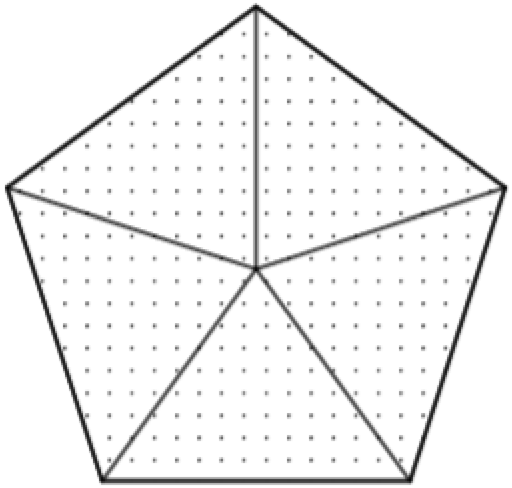


Nullity = 2



Nullity = 2^{Dim} = 4



Nullity = 1

Vectors that span the kernel can be derived from the mesh.
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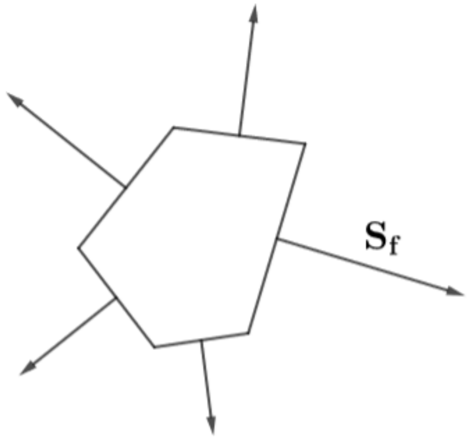
Rewriting the gradient operator

Gauss Gradient: $$G_G\phi_f = \frac{1}{V}\sum \phi_f \mathbf{S}_f$$



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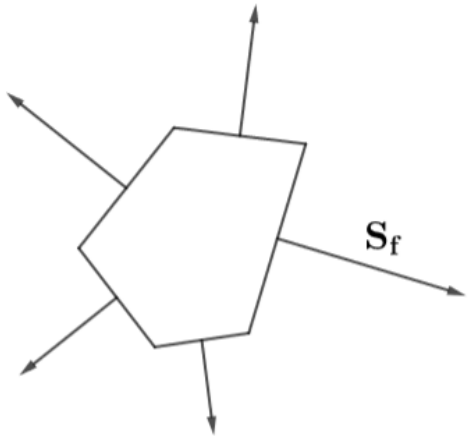

$$\sum \phi_i \mathbf{S}_f = \mathbf{0}$$

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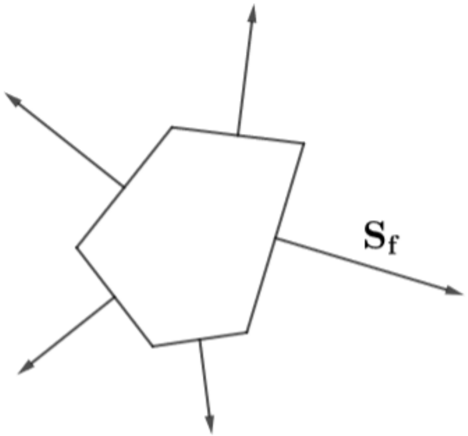
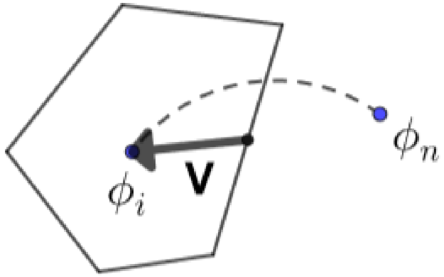


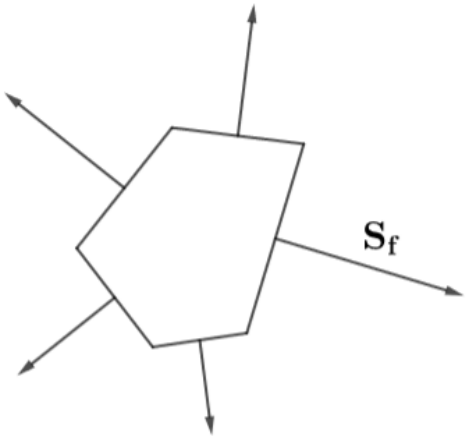
$$\sum \phi_i \mathbf{S}_f = \mathbf{0}$$

$$\Gamma_{sc}^{V} G\phi = \Omega^{-1} {\Gamma_{cs}^{V}}^{T} \Omega_s G\phi$$



Rewriting the gradient operator

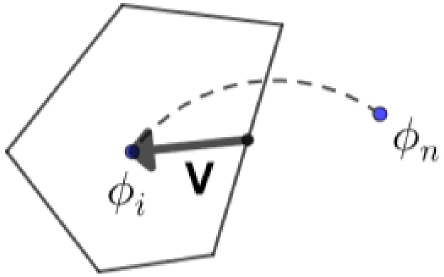
Gauss Gradient: $G_G \phi_f = \frac{1}{V} \sum \phi_f \mathbf{S}_f$



$\sum \phi_i \mathbf{S}_f = \mathbf{0}$

$$\Gamma_{sc}^{V} G\phi = \Omega^{-1} \Gamma_{cs}^{V\ T} \Omega_s G\phi$$

$$= \frac{1}{V} \sum \mathbf{S}_f d_f \begin{bmatrix} \frac{1}{d_f} w_{fi} & \\ \frac{-1}{d_f} w_{fi} & \end{bmatrix} \cdot \begin{bmatrix} \phi_n \\ \phi_i \end{bmatrix}$$



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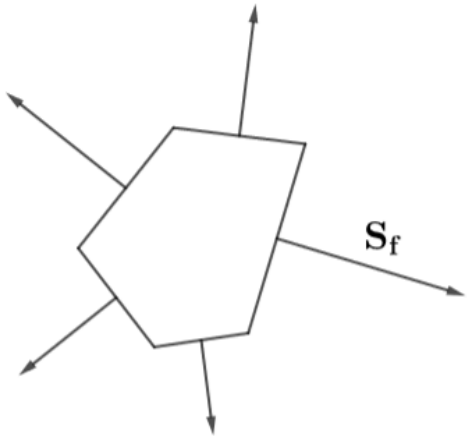

$\sum \phi_i \mathbf{S}_f = \mathbf{0}$

$$\Gamma_{sc}^{V} G\phi = \underline{\Omega}^{-1} \Gamma_{cs}^{V\ T} \Omega_s G\phi$$

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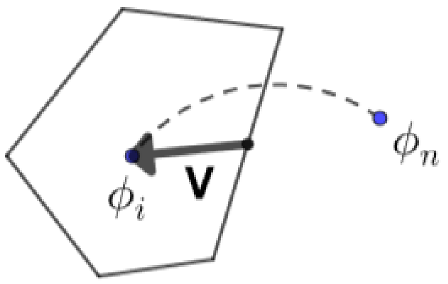

Rewriting the gradient operator

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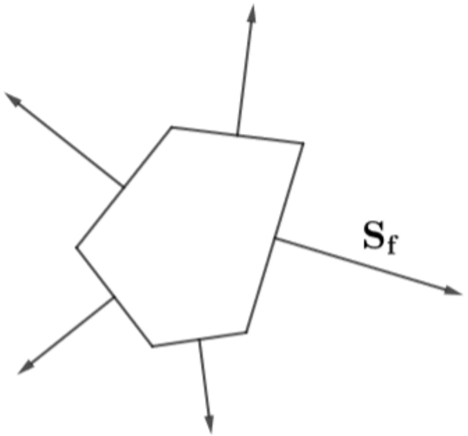

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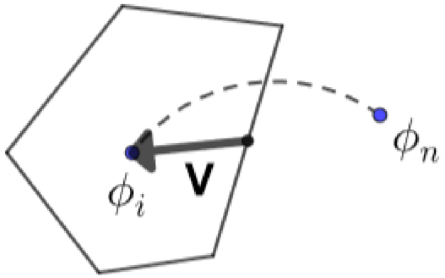

Rewriting the gradient operator

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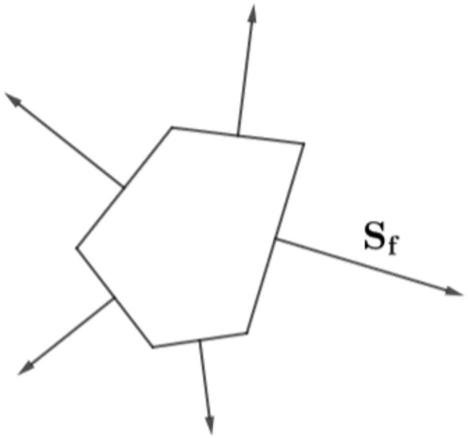

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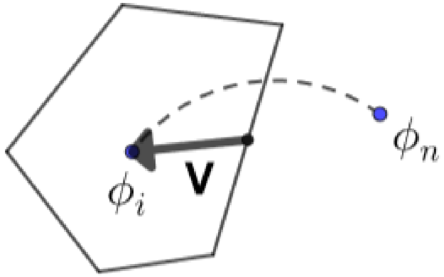

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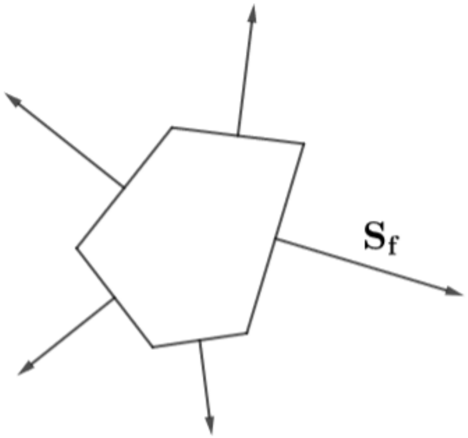

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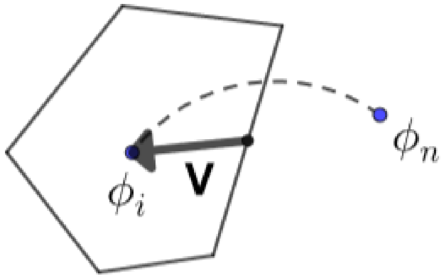

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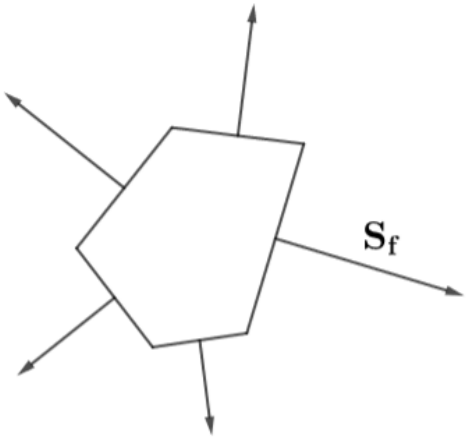

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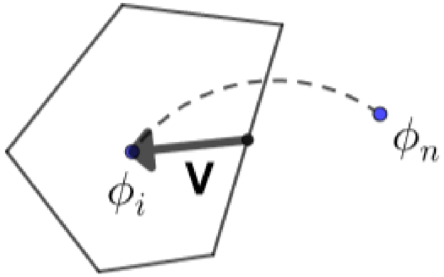
$$= \frac{1}{V}\sum \mathbf{S}_f d_f \begin{bmatrix} \frac{1}{d_f} w_{fi} \\ \frac{-1}{d_f}(1 - w_{fn}) \end{bmatrix} \cdot \begin{bmatrix} \phi_n \\ \phi_i \end{bmatrix}$$



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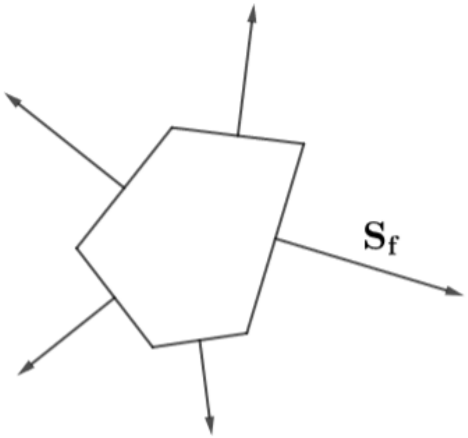

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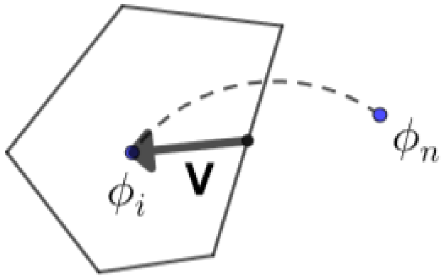

Rewriting the gradient operator

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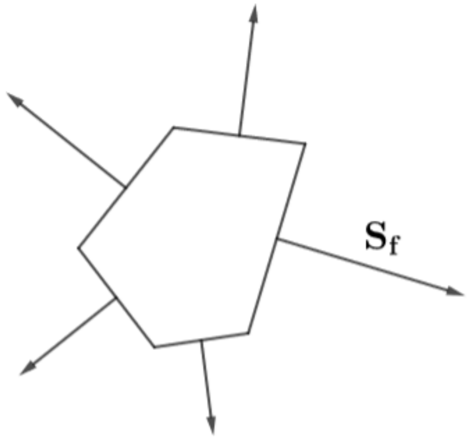

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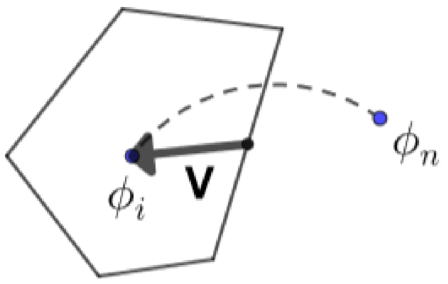

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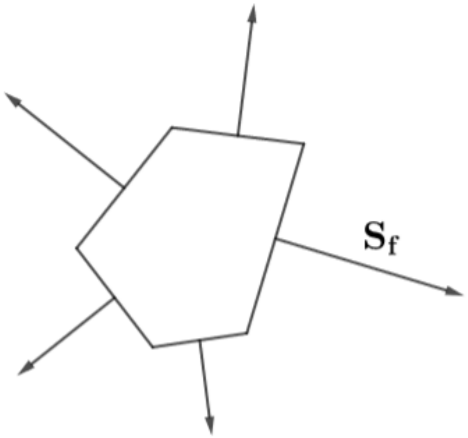

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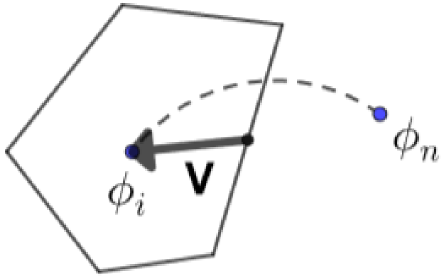
$$= G_G \Pi_{cs}^{L} \phi$$



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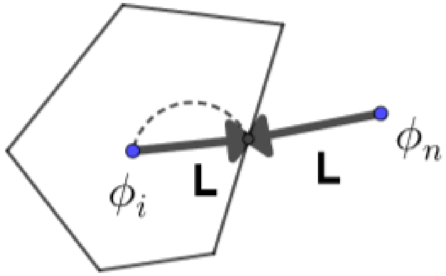
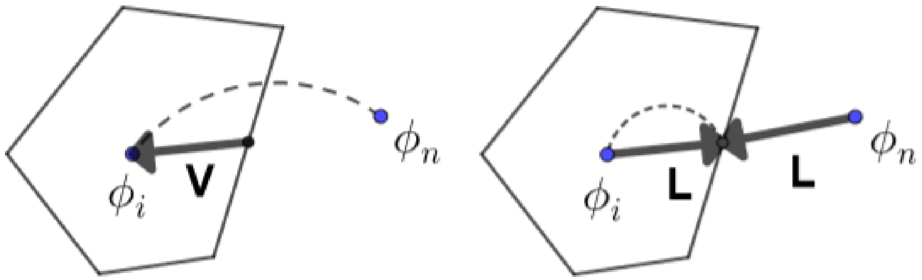


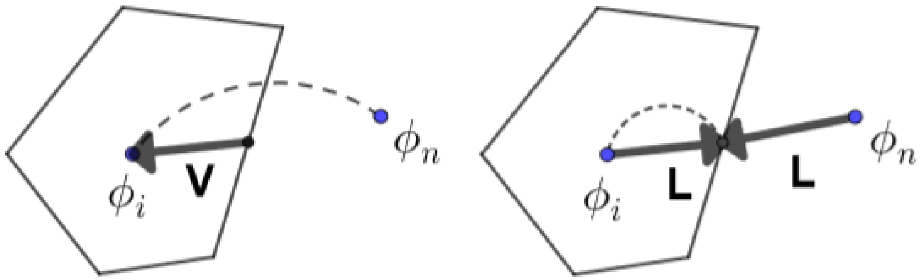
Relations between gradient operators

	$G_c = \Gamma_{sc} G$	$G_c = G_G \Pi_{cs}$
Midpoint	$\Gamma_{sc}^M G$	$G_G \Pi_{cs}^M$
Linear	$\Gamma_{sc}^L G$	$G_G \Pi_{cs}^L$
Volumetric	$\Gamma_{sc}^V G$	$G_G \Pi_{cs}^V$



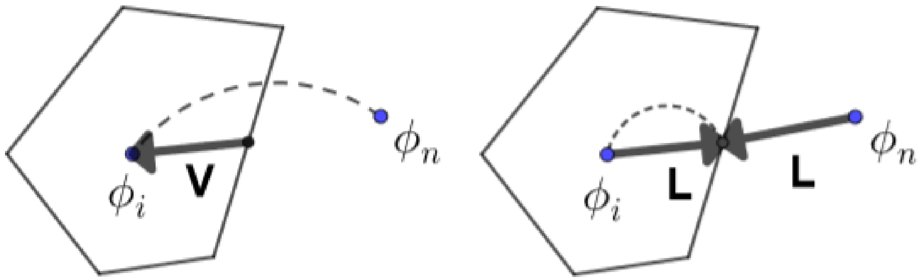
Relations between gradient operators

	$G_c = \Gamma_{sc} G$	$G_c = G_G \Pi_{cs}$
Midpoint	$\Gamma_{sc}^M G$ =	$G_G \Pi_{cs}^M$
Linear	$\Gamma_{sc}^L G$	$G_G \Pi_{cs}^L$
Volumetric	$\Gamma_{sc}^V G$	$G_G \Pi_{cs}^V$



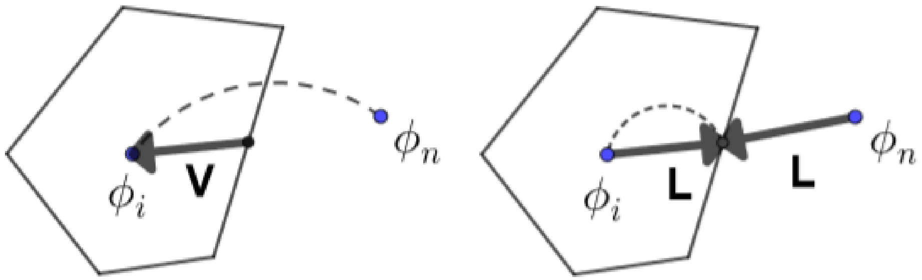
Relations between gradient operators

	$G_c = \Gamma_{sc} G$	$G_c = G_G \Pi_{cs}$
Midpoint	$\Gamma_{sc}^{M} G$ =	$G_G \Pi_{cs}^{M}$
Linear	$\Gamma_{sc}^{L} G$	$G_G \Pi_{cs}^{L}$
Volumetric	$\Gamma_{sc}^{V} G$	$G_G \Pi_{cs}^{V}$



Relations between gradient operators

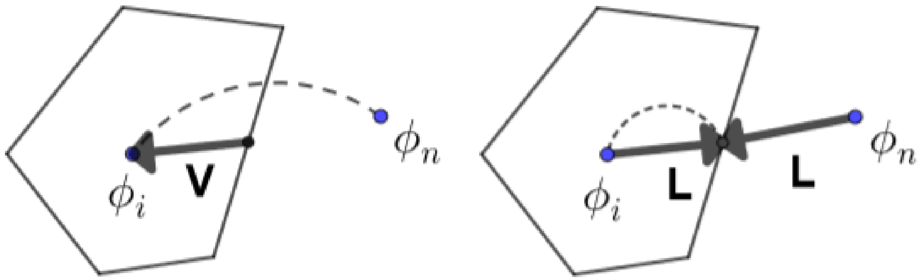
	$G_c = \Gamma_{sc} G$	$G_c = G_G \Pi_{cs}$
Midpoint	$\Gamma_{sc}^{M} G$ =	$G_G \Pi_{cs}^{M}$
Linear	$\Gamma_{sc}^{L} G$	$G_G \Pi_{cs}^{L}$
Volumetric	$\Gamma_{sc}^{V} G$	$G_G \Pi_{cs}^{V}$



Relations between gradient operators

	$G_c = \Gamma_{sc} G$	$G_c = G_G \Pi_{cs}$
Midpoint	$\Gamma_{sc}^{M} G$ =	$G_G \Pi_{cs}^{M}$
Linear	$\Gamma_{sc}^{L} G$	$G_G \Pi_{cs}^{L}$
Volumetric	$\Gamma_{sc}^{V} G$	$G_G \Pi_{cs}^{V}$

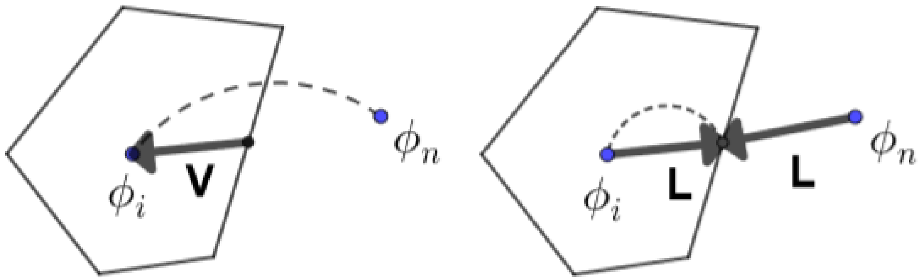
- Less options for L_c



Relations between gradient operators

	$G_c = \Gamma_{sc} G$	$G_c = G_G \Pi_{cs}$
Midpoint	$\Gamma_{sc}^M G$ =	$G_G \Pi_{cs}^M$
Linear	$\Gamma_{sc}^L G$	$G_G \Pi_{cs}^L$
Volumetric	$\Gamma_{sc}^V G$	$G_G \Pi_{cs}^V$

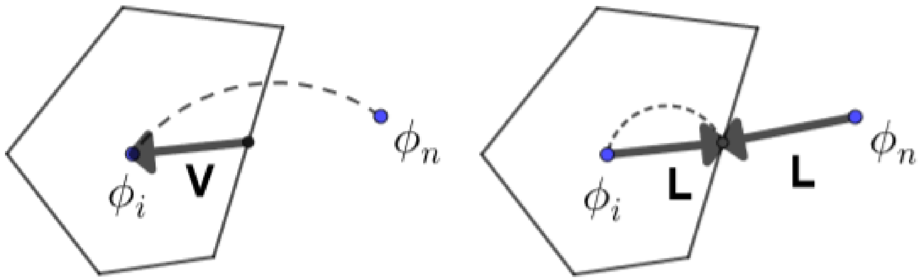
- Less options for L_c
- G_G is easier to implement



Relations between gradient operators

	$G_c = \Gamma_{sc}G$	$G_c = G_G\Pi_{cs}$
Midpoint	$\Gamma_{sc}^{M}G$	$G_G\Pi_{cs}^{M}$
Linear	$\Gamma_{sc}^{L}G$	$G_G\Pi_{cs}^{L}$
Volumetric	$\Gamma_{sc}^{V}G$	$G_G\Pi_{cs}^{V}$

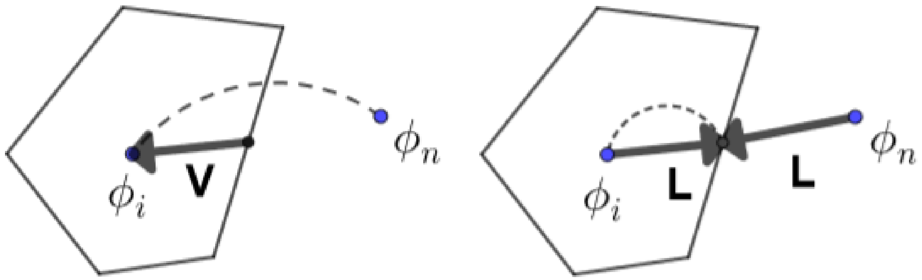
- Less options for L_c
- G_G is easier to implement
- Many solvers use $M\Gamma_{cs}^{L}G_G\Pi_{cs}^{L}$ which is non-symmetric



Relations between gradient operators

	$G_c = \Gamma_{sc} G$	$G_c = G_G \Pi_{cs}$
Midpoint	$\Gamma_{sc}^M G$ =	$G_G \Pi_{cs}^M$
Linear	$\Gamma_{sc}^L G$	$G_G \Pi_{cs}^L$
Volumetric	$\Gamma_{sc}^V G$	$G_G \Pi_{cs}^V$

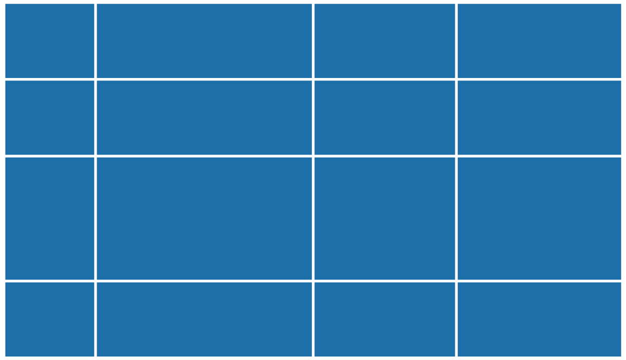
- Less options for L_c
- G_G is easier to implement
- Many solvers use $M\Gamma_{cs}^L G_G \Pi_{cs}^L$ which is non-symmetric
- Useful in deriving kernel vectors



Predicting kernel vectors

Predicting kernel vectors

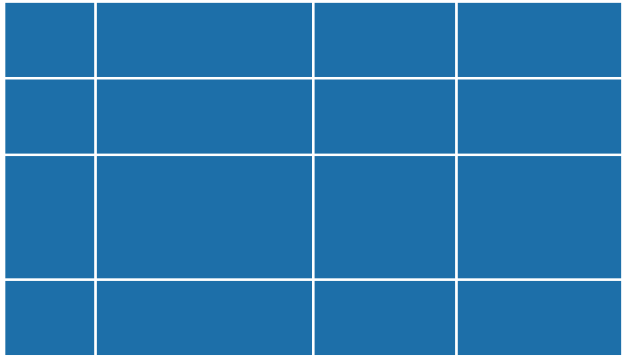
$$[\mathbf{p}_c^{-(0)}]_{i,j} = 1$$

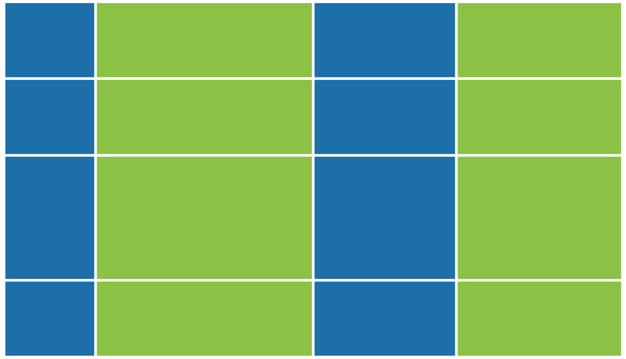


Predicting kernel vectors

$$[\mathbf{p}_c^{-(0)}]_{i,j} = 1$$

$$[\mathbf{p}_c^{-(1)}]_{i,j} = (-1)^i$$



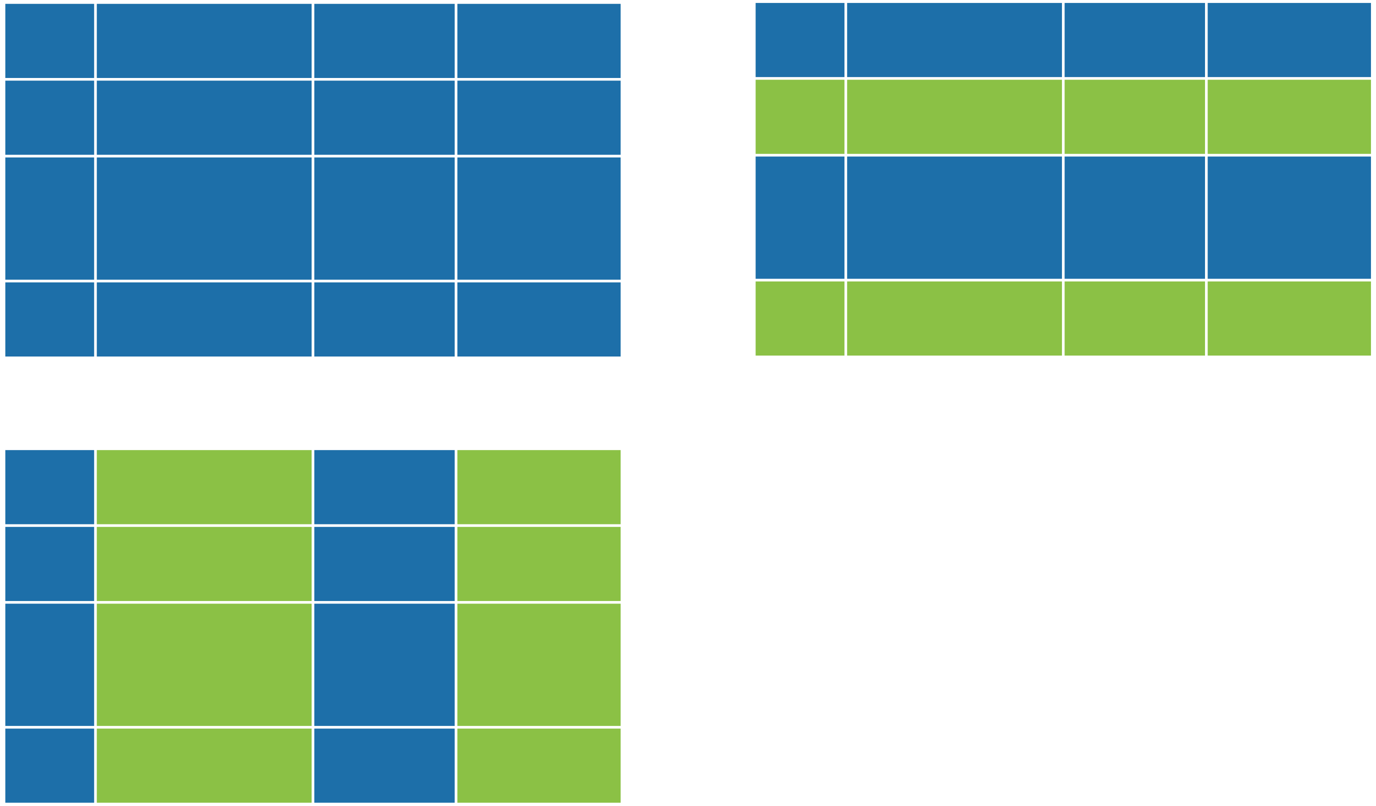


Predicting kernel vectors

$$[\mathbf{p}_c^{-(0)}]_{i,j} = 1$$

$$[\mathbf{p}_c^{-(1)}]_{i,j} = (-1)^i$$

$$[\mathbf{p}_c^{-(2)}]_{i,j} = (-1)^j$$



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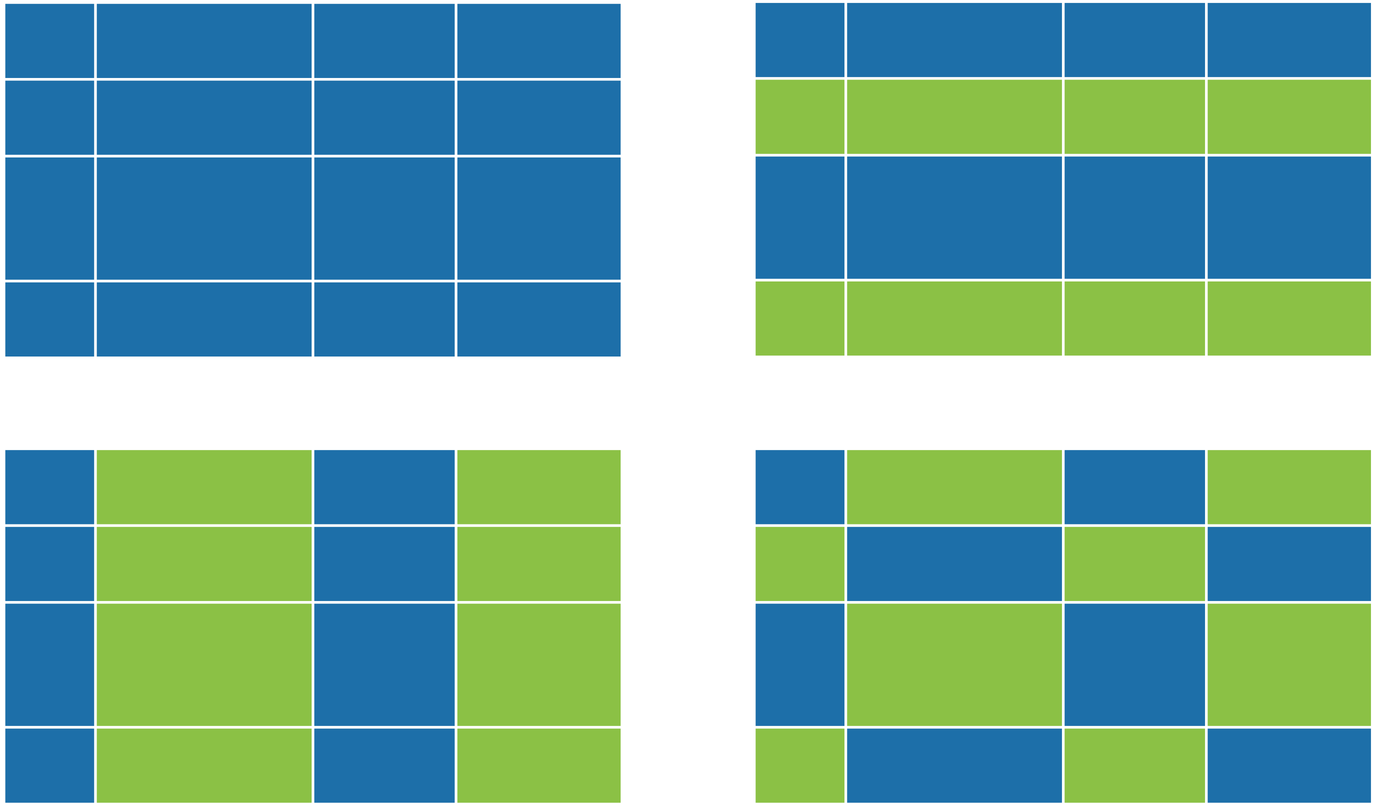
$$[\mathbf{p}_c^{-(2)}]_{i,j} = (-1)^j$$

$$[\mathbf{p}_c^{-(3)}]_{i,j} = (-1)^{i+j}$$



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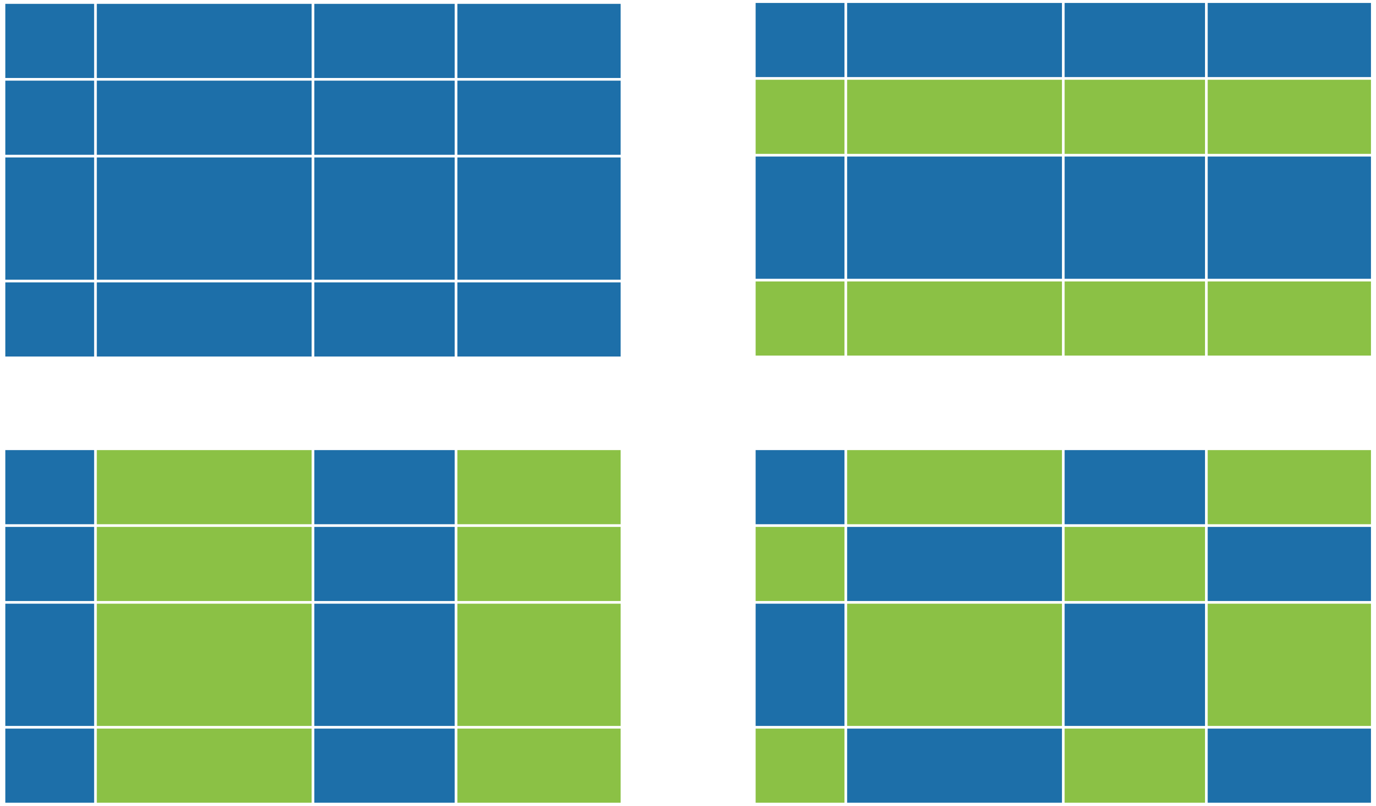

$$G_c = G_G \Pi_{cs}^{L}: \quad \mathbf{p}_c^{-(1)} = \Delta x_0 \quad -\Delta x_1 \quad \Delta x_2 \quad -\Delta x_3$$

Predicting kernel vectors

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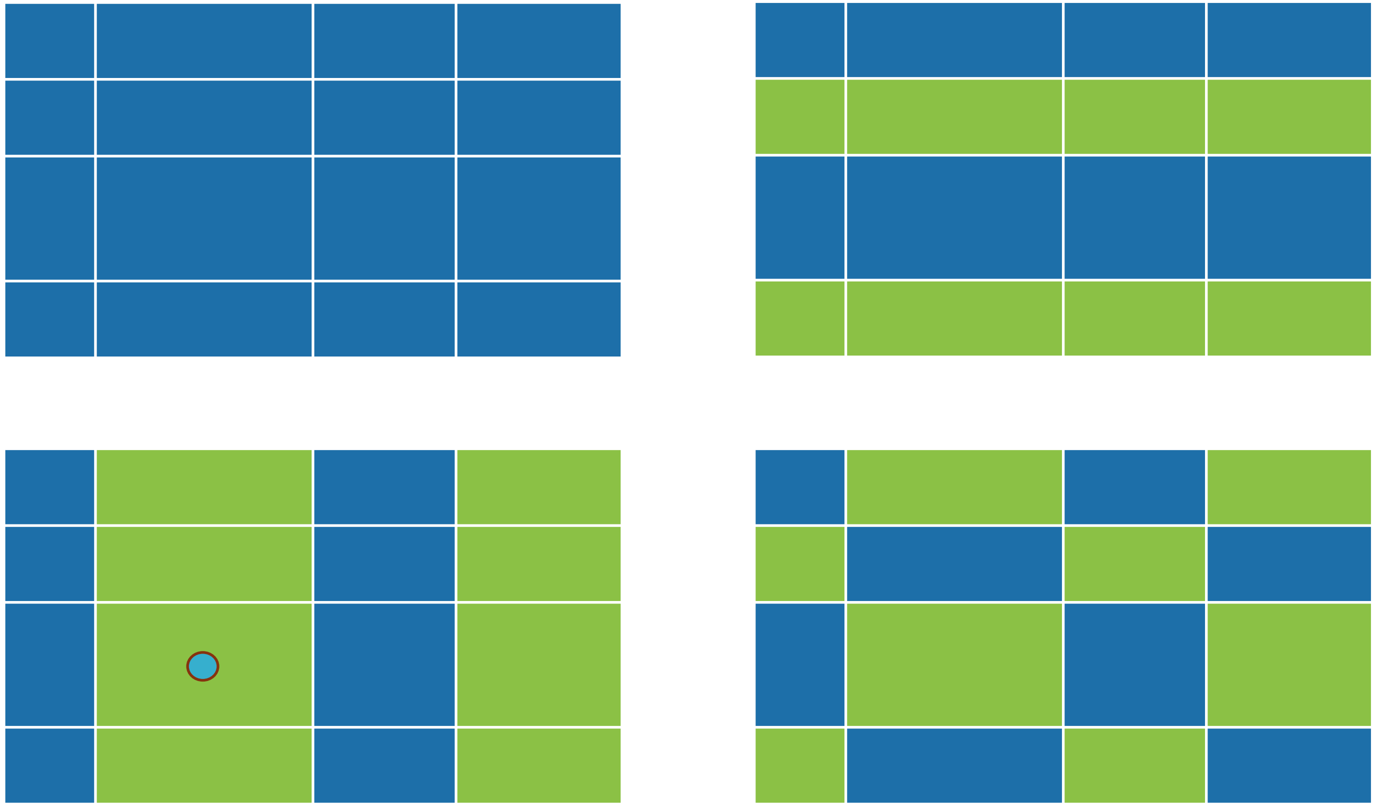

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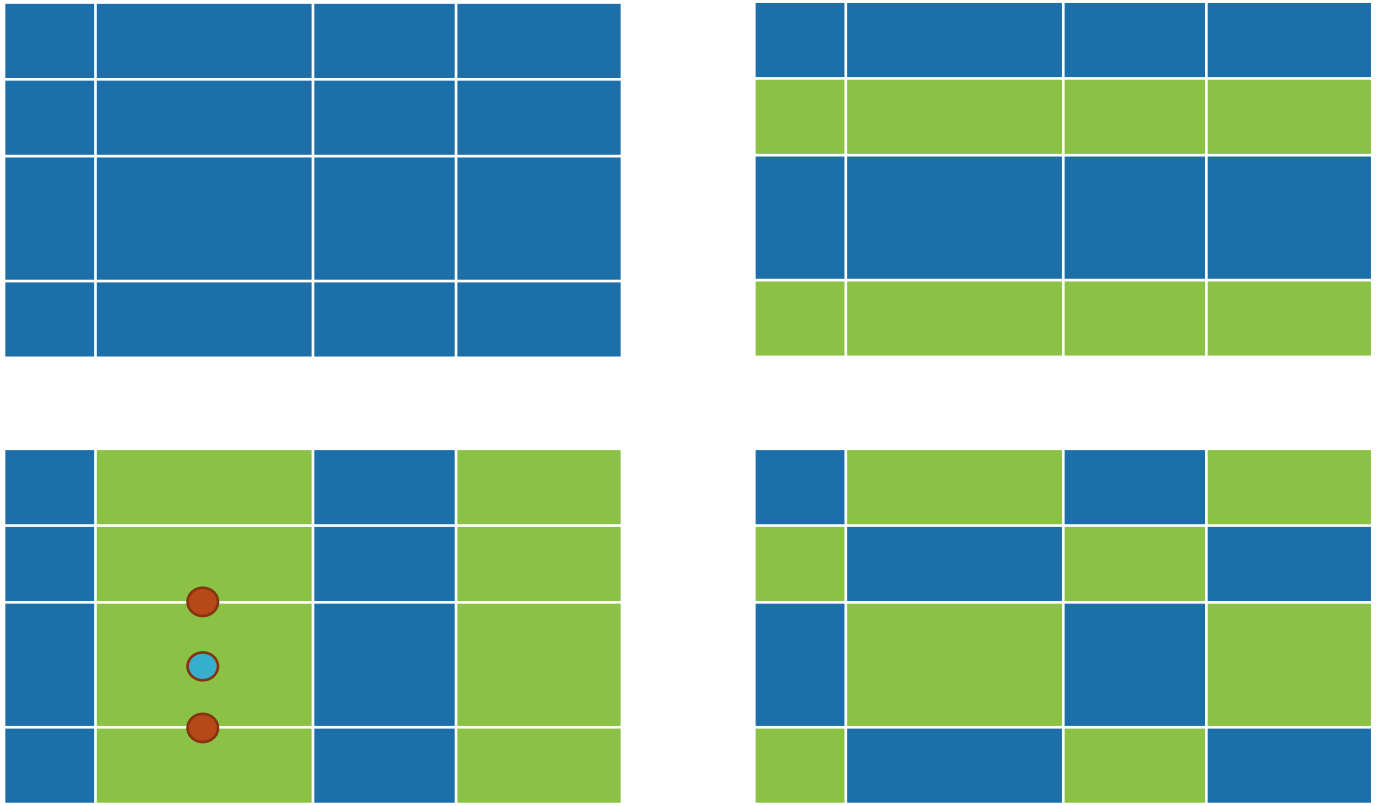

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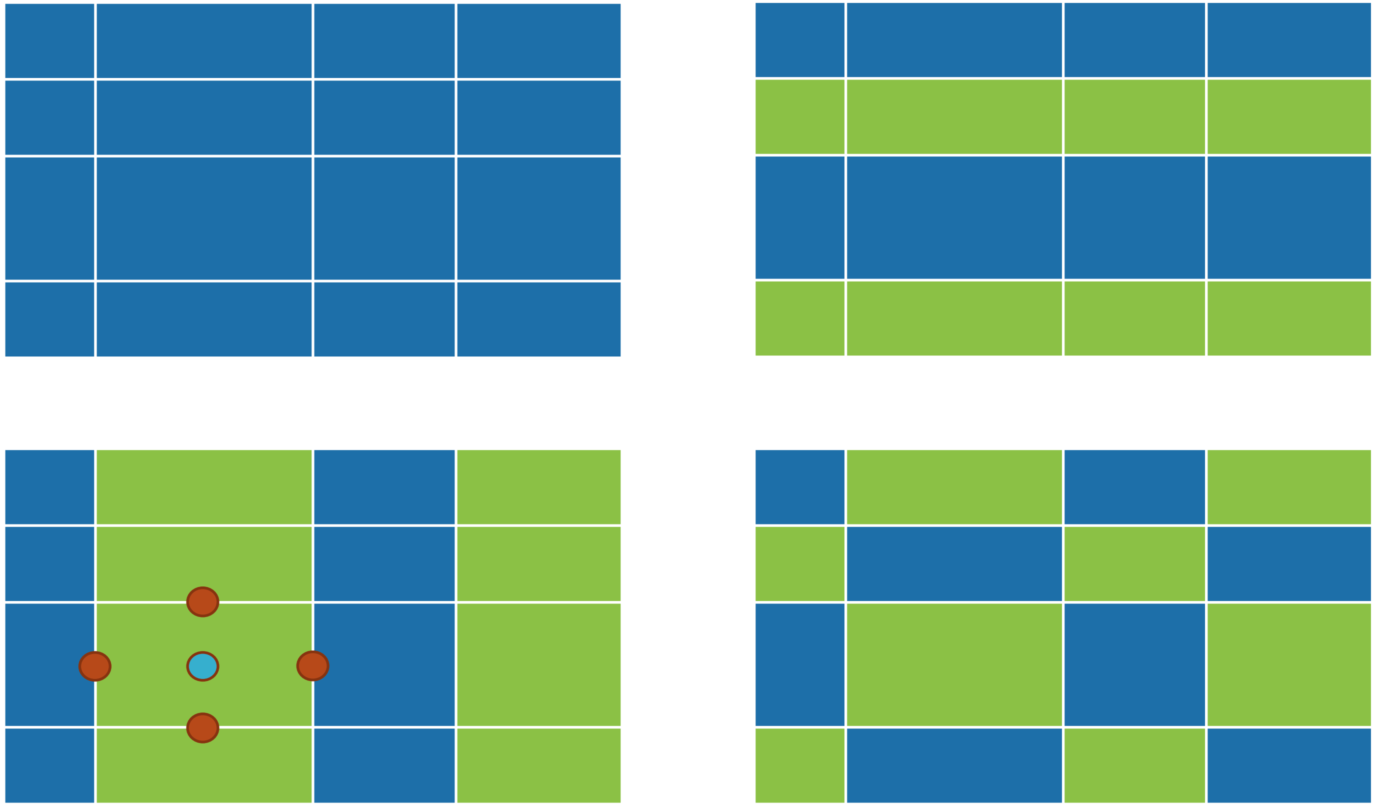


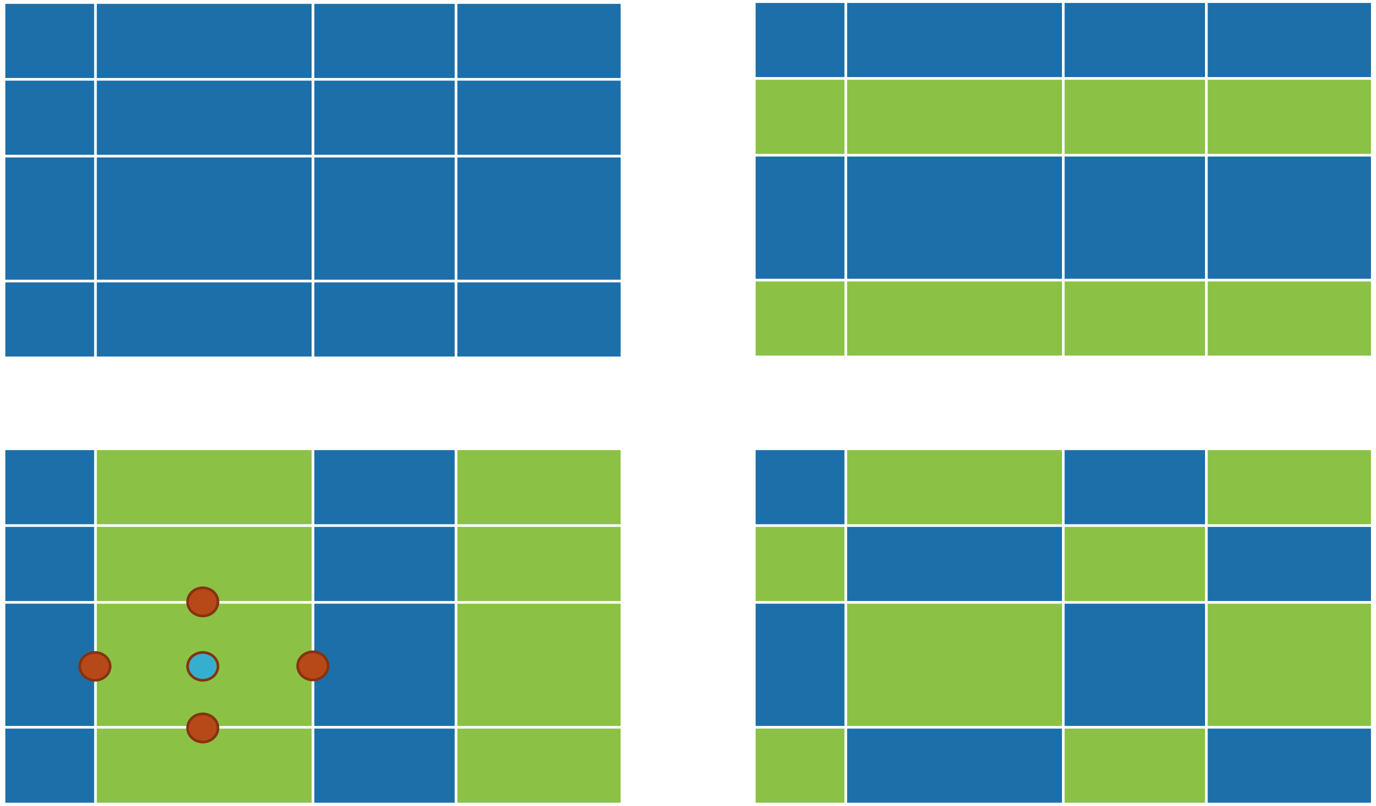
$$G_c = G_G \Pi_{cs}^{L}: \quad \mathbf{p}_c^{-(1)} = \Delta x_0 \quad -\Delta x_1 \quad \Delta x_2 \quad -\Delta x_3$$

$$\Pi_{cs}^{L} \rightarrow \phi_w = \frac{\Delta x_0 \Delta x_1 - \Delta x_1 \Delta x_0}{\Delta x_0 + \Delta x_1} = 0$$

Predicting kernel vectors

$$[\mathbf{p}_c^{-(0)}]_{i,j} = 1$$
$$[\mathbf{p}_c^{-(1)}]_{i,j} = (-1)^i (\Delta x_i)^\alpha$$
$$[\mathbf{p}_c^{-(2)}]_{i,j} = (-1)^j (\Delta y_j)^\alpha$$
$$[\mathbf{p}_c^{-(3)}]_{i,j} = (-1)^{i+j} (\Delta x_i \Delta y_j)^\alpha$$

$$\alpha = \begin{cases} 1, & \text{if linear} \\ 0, & \text{if midpoint} \\ -1, & \text{if volumetric} \end{cases}$$



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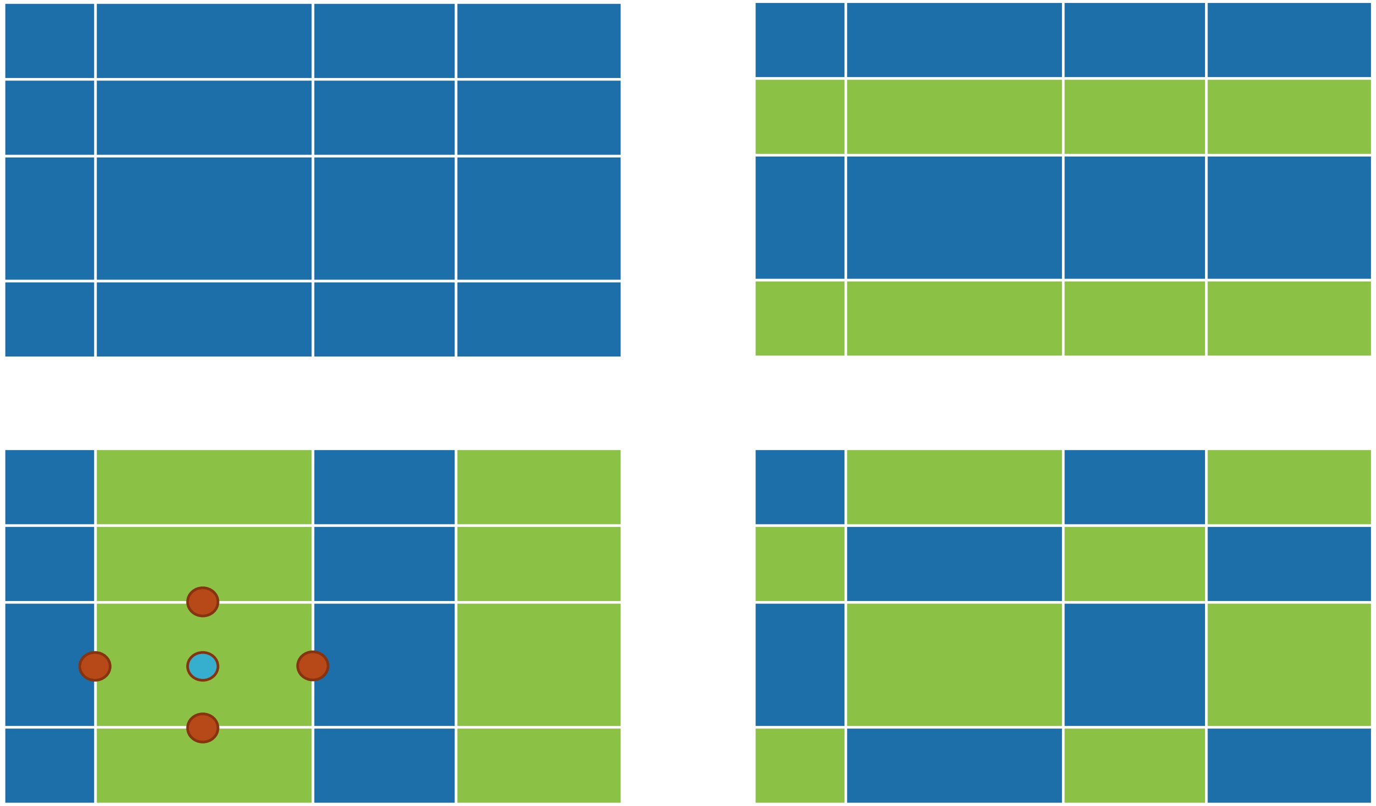
$$[\mathbf{p}_c^{-(3)}]_{i,j} = (-1)^{i+j} (\Delta x_i \Delta y_j)^\alpha$$

$$\alpha = \begin{cases} 1, & \text{if linear} \\ 0, & \text{if midpoint} \\ -1, & \text{if volumetric} \end{cases}$$



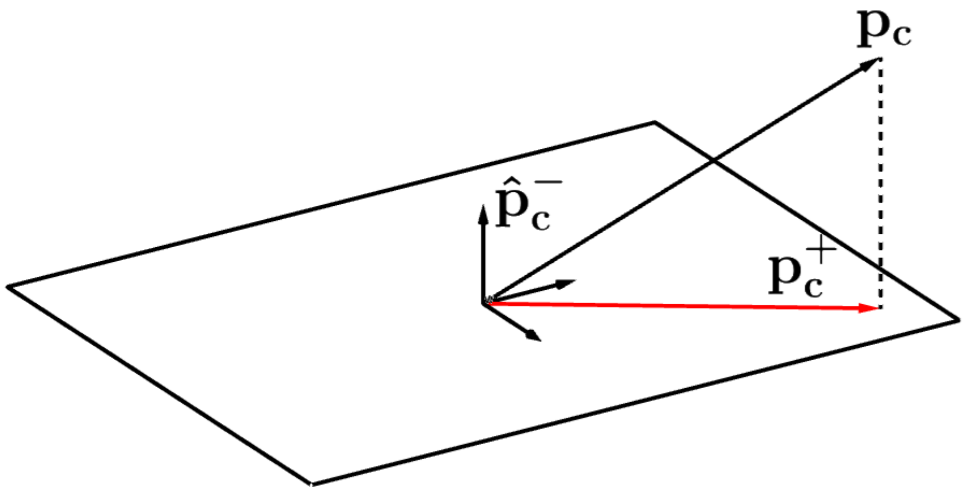
$$G_c = G_G \Pi_{cs}^L : \quad \mathbf{p}_c^{-(1)} = \Delta x_0 \quad -\Delta x_1 \quad \Delta x_2 \quad -\Delta x_3$$

$$\Pi_{cs}^L \to \phi_w = \frac{\Delta x_0 \Delta x_1 - \Delta x_1 \Delta x_0}{\Delta x_0 + \Delta x_1} = 0$$

Although not necessarily orthogonal, they are linearly independent, spanning the nullspace of L_c

Filtering spurious modes

$$\mathbf{p}_c^+ = \mathbf{p}_c - \sum_i \left(\mathbf{p}_c \cdot \hat{\mathbf{p}}_c^{-(i)} \right) \hat{\mathbf{p}}_c^{-(i)}$$



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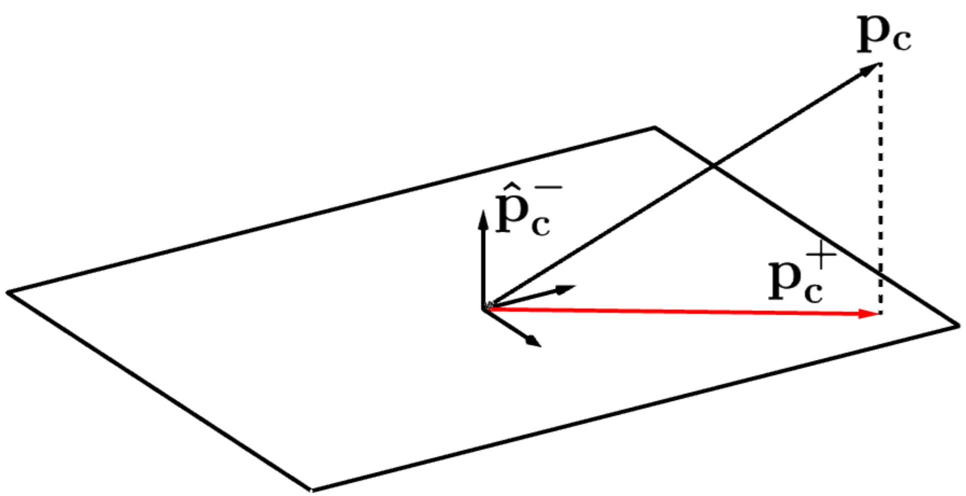

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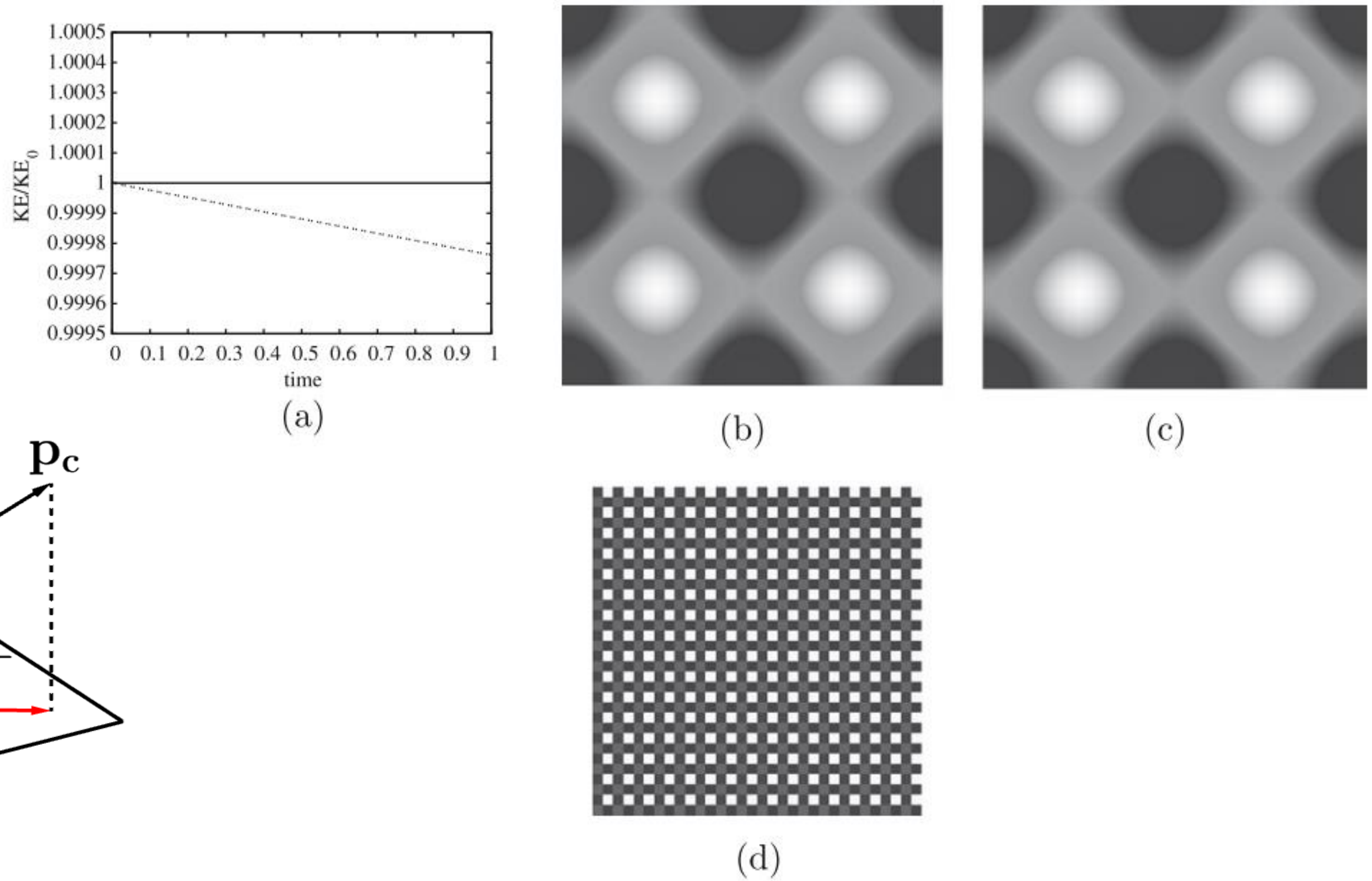

Fig. 1. Inviscid Taylor vortex. (a) Temporal evolution of kinetic energy using the present (solid) and Rhie–Chow (dotted) methods. (b–d) Pressure contours for the present (b), Rhie–Chow method (c) and without any correction (d).

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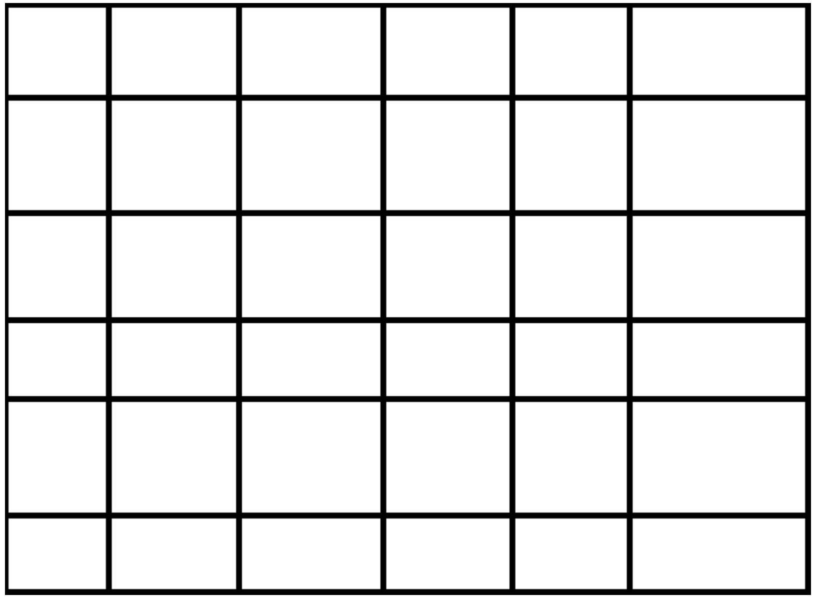

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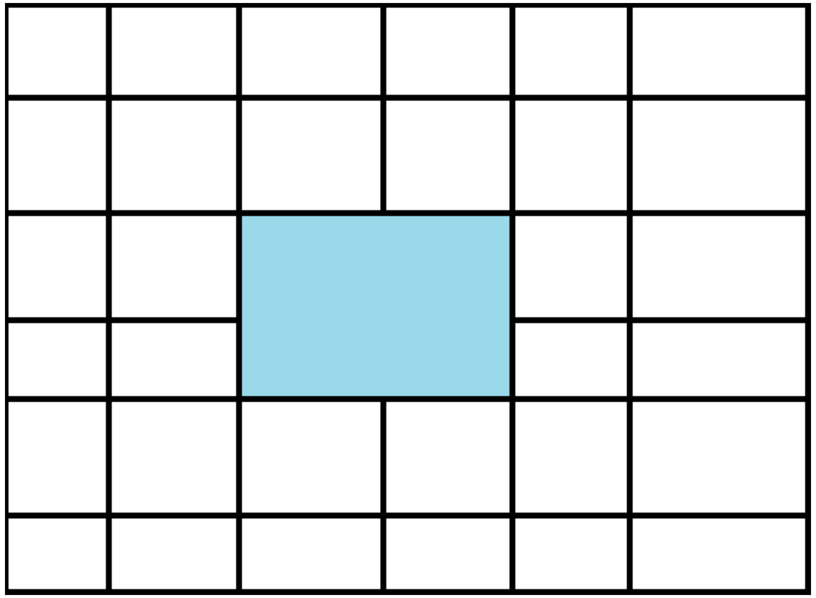

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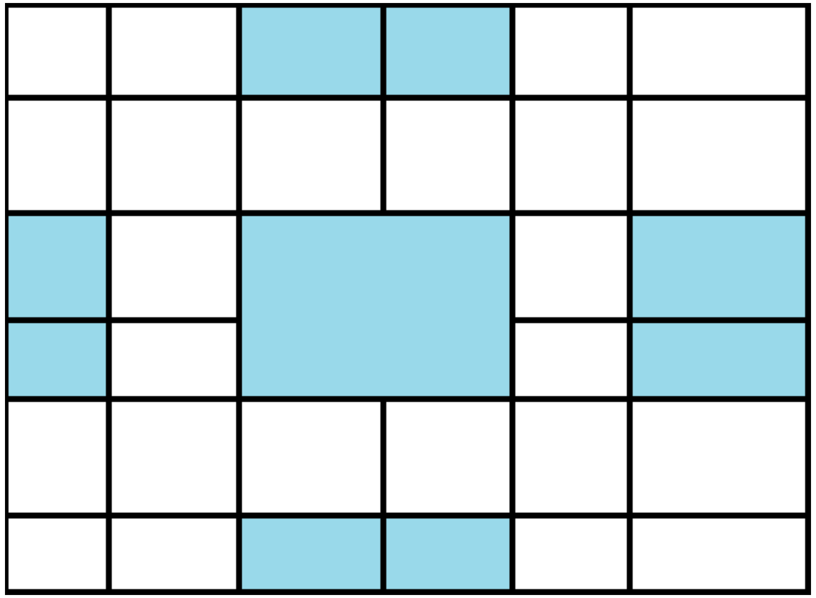

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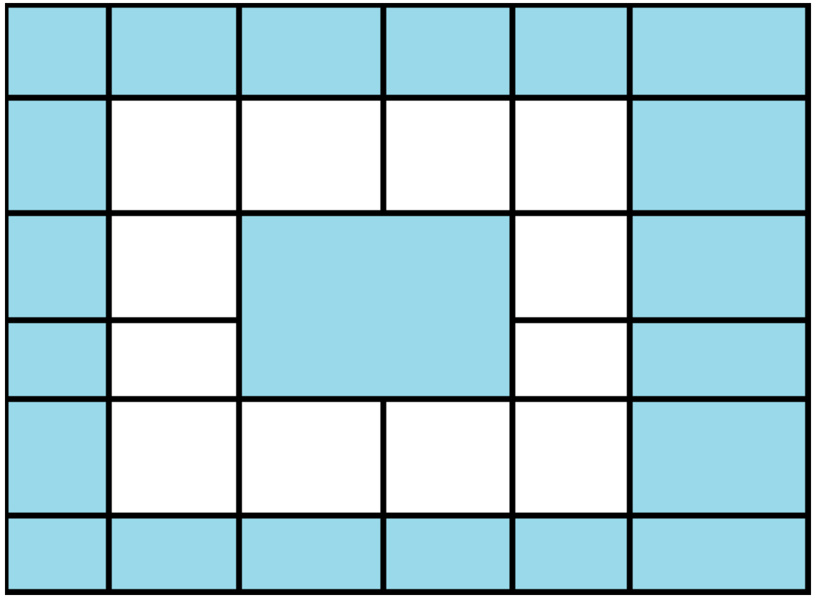

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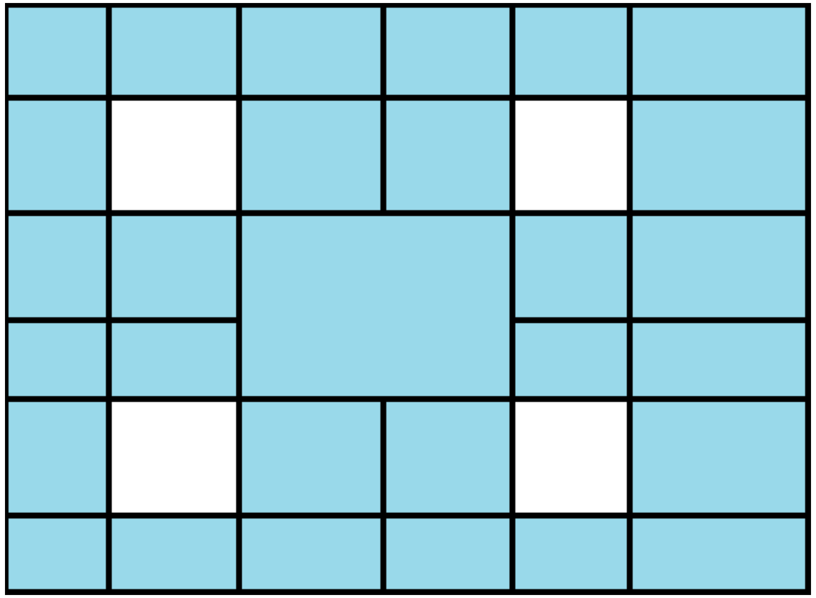

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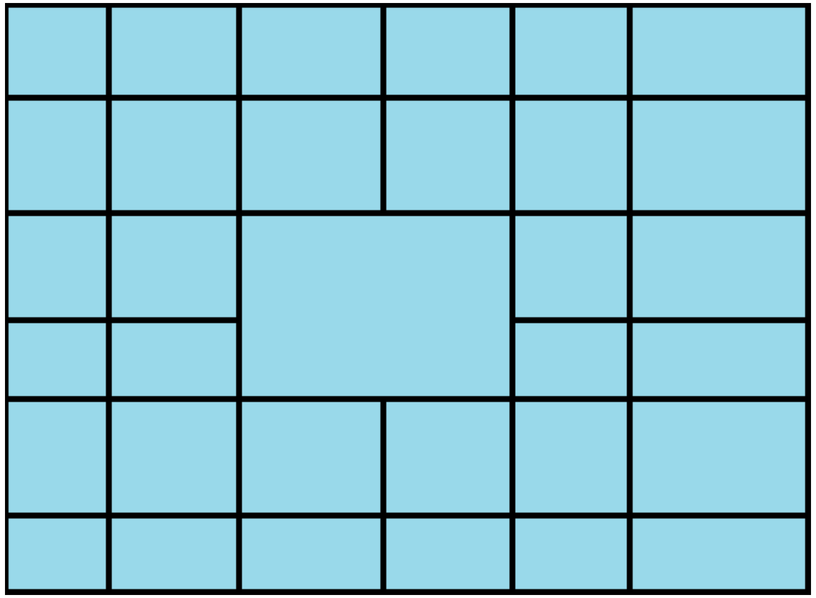

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How do spurious modes arise?

- Non-symmetry of Laplacian operator?
- Solver?
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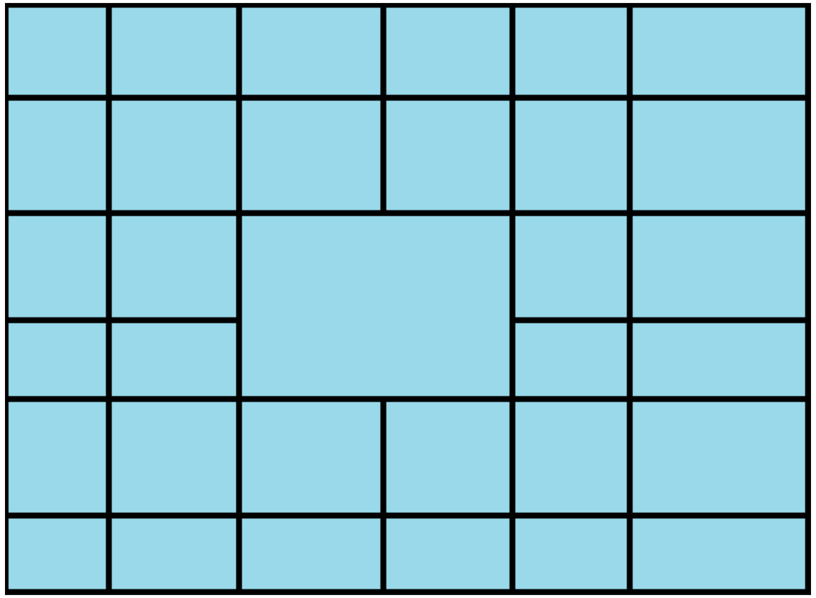
Simple mesh changes can reduce nullity to 1.

- Will this eliminate checkerboarding?



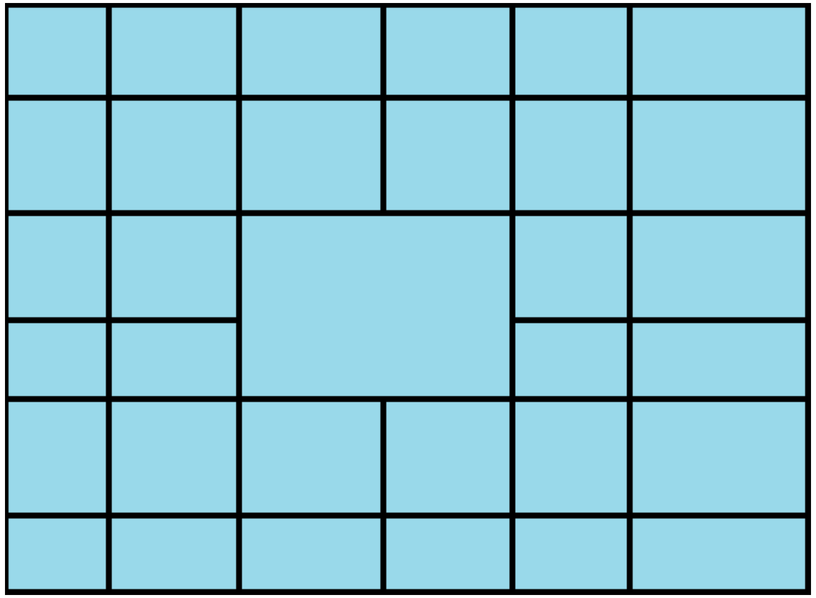
Discussion

How do spurious modes arise?

- Non-symmetry of Laplacian operator?
- Solver?
- Rounding errors?

Simple mesh changes can reduce nullity to 1.

- Will this eliminate checkerboarding?
- Are other (low EV) modes also problematic?



Thank you for attending!