An energy-preserving unconditionally stable fractional step method for DNS/LES on collocated grids

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1. Motivation of this work

Motivation: Is it possible to find an energy-preserving unconditionally stable fractional step method on collocated grids for any mesh? Instead of changing randomly the numerical schemes, the mesh or...

2. Definition of basic collocated operators

Let us suppose we have n control volumes and m faces.

Finite volume discretization of incompressible NS equations on an arbitrary collocated mesh

$$\Omega \frac{d\mathbf{u}_c}{dt} + C(\mathbf{u}_s)\mathbf{u}_c = -D\mathbf{u}_c - \Omega G_c \mathbf{p}_c, \quad (1)$$

$$M\mathbf{u}_s = \mathbf{0}_c. \quad (2)$$

- $\mathbf{p}_c = (p_1, ..., p_n)^T \in \mathbb{R}^n$ is the cell-centered pressure.
- $\mathbf{u}_c = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^T \in \mathbb{R}^{3n}$, where $\mathbf{u}_i = ((u_i)_1, ..., (u_i)_n)^T$ are the vectors containing the velocity components corresponding to the x_i-spatial direction.
- $\mathbf{u}_s = ((u_s)_1, ..., (u_s)_m)^T \in \mathbb{R}^m$ is the staggered velocity.
- The velocities are related via the interpolator from cells to faces $\Gamma_{c \to s} \in \mathbb{R}^{m \times 3n} \implies \mathbf{u}_s = \Gamma_{c \to s}\mathbf{u}_c$.

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3. Conservation of global kinetic energy

Global kinetic energy equation

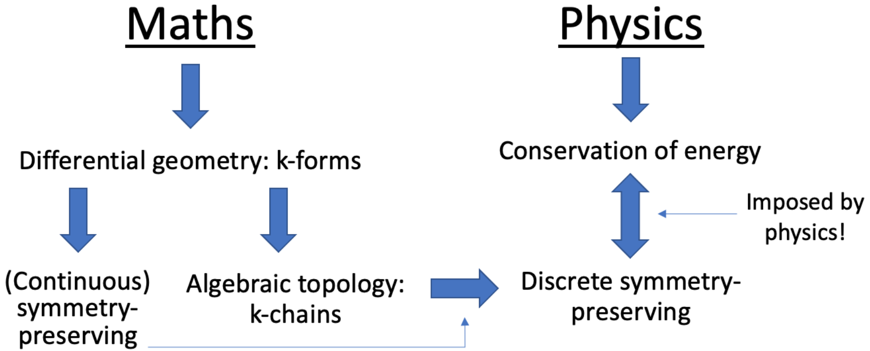
$$\frac{d||\mathbf{u}_c||^2}{dt} = -\mathbf{u}_c^T(C(\mathbf{u}_s) + C^T(\mathbf{u}_s))\mathbf{u}_c - \mathbf{u}_c^T(D + D^T)\mathbf{u}_c - \mathbf{u}_c^T \Omega G_c \mathbf{p}_c - \mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c. \quad (3)$$

In absence of diffusion, that is $D = 0$, the global kinetic energy is conserved if:

- $C(\mathbf{u}_s) = -C^T(\mathbf{u}_s)$, i.e, the convective operator should be skew-symmetric.
- $(-\Omega G_c)^T = M\Gamma_{c\rightarrow s}$, because $M\mathbf{u}_s = \mathbf{0}_c$.

Question: Can we find a mathematical reason to justify these relations, instead of a physical one?

Mimicking continuos properties



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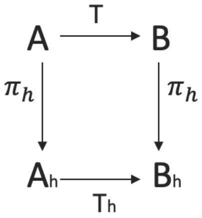


- A and B are two vectorial spaces.
- π_h is a discretization operator.
- T is a continuous operator.
- A_h, B_h and T_h are the discrete counterparts of A, B and T, respectively.

We will require this diagram to be commutative.

4. A stable pressure gradient interpolation

FSM iterative Poisson equation in collocated meshes

$$L\tilde{p}_c^{n+1} = M_c u_c^n \quad \longrightarrow \quad u_c^{n+1} = u_c^n - G_c \tilde{p}_c^{n+1}, \tag{4}$$

where $M_c = M\Gamma_{c\rightarrow s}$and $G_c = \Gamma_{s\rightarrow c} G = -\Omega_c^{-1}\Gamma_{c\rightarrow s}^T M^T$ are the collocated divergence and the collocated gradient. Developing the correction in u_c^n:

$$u_c^n = u_c^{n-1} - G_c\tilde{p}_c^n = u_c^{n-2} - G_c\tilde{p}_c^n - G_c\tilde{p}_c^{n-1} = \cdots = u_c^p - G_c \sum_{i=1}^{n} \tilde{p}_c^i \tag{5}$$

So, the acumulated pressure at n iteration is:

$$p_c^n = \sum_{i=1}^{n} \tilde{p}_c^i \tag{6}$$

Introducing all this in (7) we obtain:

$$Lp_c^{n+1} = M\Gamma_{c\rightarrow s} u_c^p + (L - L_c)p_c^n. \tag{7}$$

- **Mid-point scheme:** $\phi_f = \frac{1}{2}(\phi_{c1} + \phi_{c2})$.
- **Volume weighted scheme:** $\phi_f = \frac{V_{s1}}{V_{s1}+V_{s2}}\phi_{c1} + \frac{V_{s2}}{V_{s1}+V_{s2}}\phi_{c2}$.

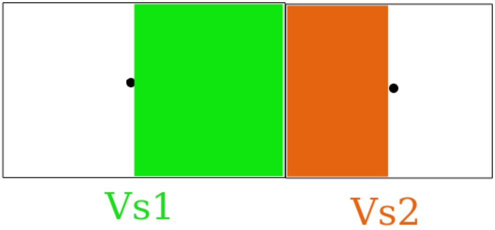


Figure 1: Volume weighted volumes

A stable pressure gradient interpolation

- Stable solutions → Eigenvalues of $L - L_c$ negative.
- This can be achieved by using the volume weighted scheme:

$$\Pi_{c \to s} = \Delta_s^{-1} \Delta_{sc}^T, \tag{8}$$

where $\Delta_s \in \mathbb{R}^{m \times m}$ is a diagonal matrix containing the projected distances between two adjacent control volumes, and $\Delta_{sc} \in \mathbb{R}^{m \times n}$ is a matrix containing the projected distances between an adjacent cell node and its corresponding face.

This problem was widely adressed in: *D. Santos, J. Muela, N. Valle, F.X. Trias, On the Interpolation Problem for the Poisson Equation on Collocated Meshes. 14th WCCM-ECCOMAS Congress 2020, DOI: 10.23967/wccm-eccomas.2020.257.*

5. Test case: Turbulent channel flow $Re_\tau = 395$

In order to check the stability and accuracy of the method, some tests have been carried out:

- $Re_\tau = 395$.
- y^+ around 1.
- Domain $5\pi x 2 x \pi$, periodic x and z directions.

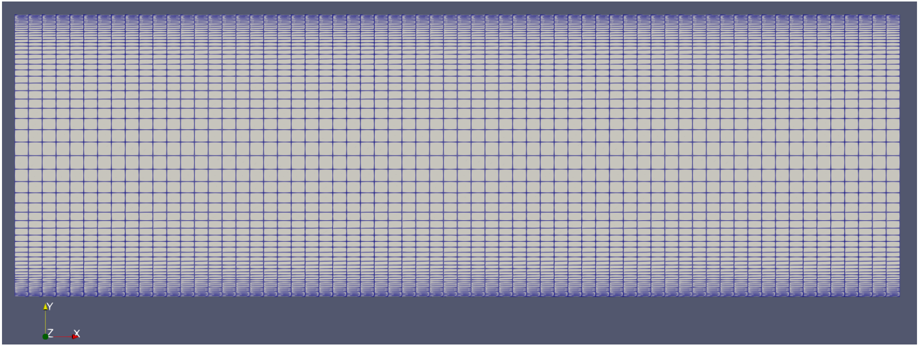
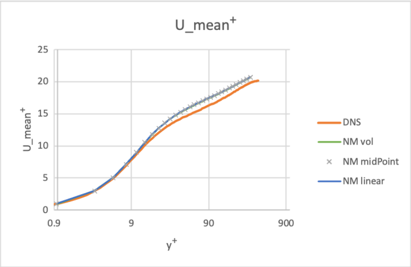
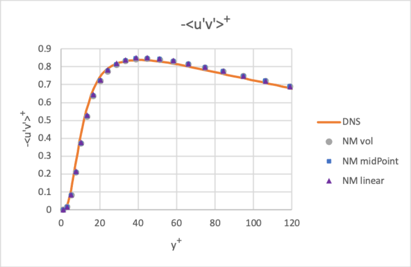
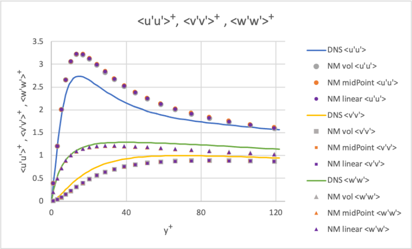


Figure 2: 64x64x64 mesh used.

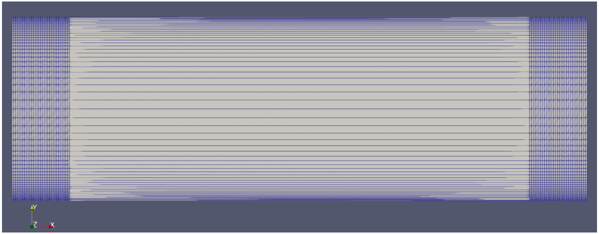
Test case: Turbulent channel flow $Re_\tau = 395$.

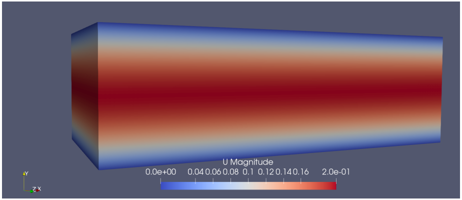






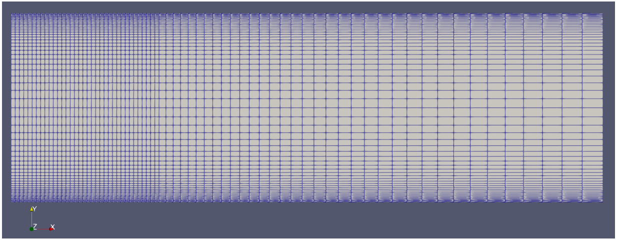
Turbulent channel flow $Re_\tau = 395$. High distorted mesh.

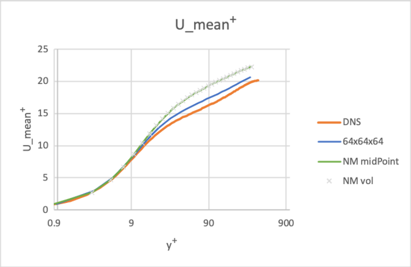


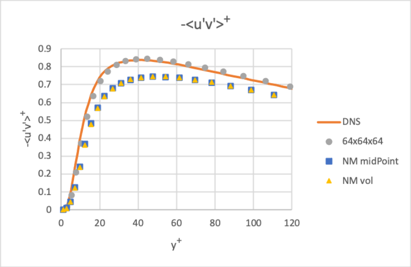


Pressure gradient interpolated using a volume weighted interpolator. MidPoint or linear are blowing up the simulation.

Turbulent channel flow $Re_\tau = 395$. Progressive mesh

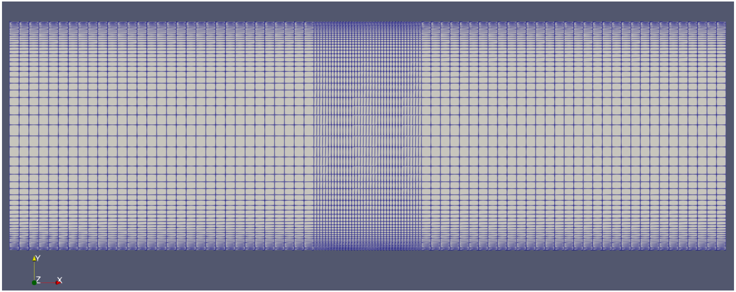






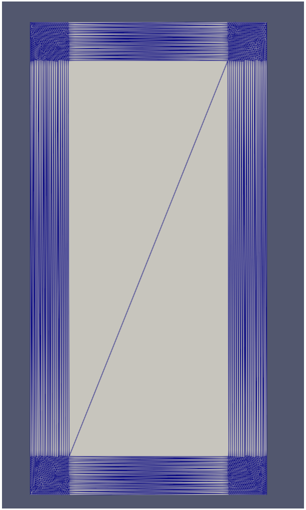
Linear doesn't trigger turbulence.

Turbulent channel flow $Re_\tau = 395$. Center refinement

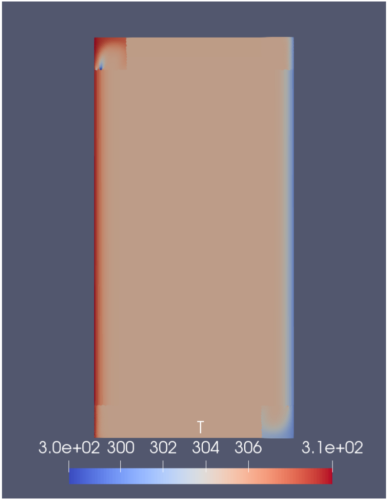


The three interpolators give the same (reasonable) results.

Preliminary results on unstructured meshes.



Highly distroted mesh.



Temperature distribution.

Accuracy and stability conclusions:

- Three interpolators have been tested for the interpolation of the pressure gradient from faces to cells: volume weighted, linear and midPoint interpolations.
- All three seem to have the same accuracy in high quality meshes.
- When distorting the mesh, the first one that looses accuracy and eventually blows up is the linear interpolation.
- The midPoint interpolation seems to be more stable than the linear, but it is still blowing up in highly distorted meshes.
- Volumetric interpolation is an unconditionally stable interpolation.

Conclusions

General conclusions

- An energy-preserving unconditionally stable fractional step method on collocated grids has been presented.
- There are mathematical reasons beyond physical ones in order to preserve the underlying symmetries of the differential operators.
- The appearance of unphysical velocities is a common problem found in highly distorted meshes, and it solved by means of interpolating the pressure gradient using a volumetric scheme.

Future work:

- Is there a moment when the midPoint interpolation blows up and the volumetric still gives good results?
- Test accuracy on unstructured meshes.
- Test the accuracy of the solution when varying progressively the distortion of the mesh.