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MESH CONSTRAINTS FOR AN ENERGY PRESERVING UNCONDITIONALLY STABLE PROJECTION METHOD ON COLLOCATED UNSTRUCTURED GRIDS

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INTRODUCTION

The Navier-Stokes equations for Newtonian, incompressible flows in dimensionless primitive variables read:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = \frac{1}{Re}\Delta\boldsymbol{u} - \nabla p, \tag{1a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{1b}$$

where Re represents the dimensionless Reynolds number. A fully-conservative finite-volume discretization, maintaining the symmetries of the differential operators for collocated unstructured meshes, was firstly introduced in [1]. Assuming there are n control volumes and m faces:

$$\Omega\frac{d\boldsymbol{u}_c}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_c + \mathsf{D}\boldsymbol{u}_c + \Omega\mathsf{G}_c\boldsymbol{p}_c = \boldsymbol{0}_c, \tag{2a}$$

$$\mathsf{M}\boldsymbol{u}_s = \boldsymbol{0}_c, \tag{2b}$$

where $\boldsymbol{p}_c = (p_1, p_2, \ldots, p_n)^T \in \mathbb{R}^n$ and $\boldsymbol{u}_c \in \mathbb{R}^{3n}$ are the cell-centered pressure and collocated velocity fields, respectively. The subindices c and s indicate if the variables are cell-centered or staggered at the faces. To verify mass conservation within each control volume, a velocity field is defined at the faces $\boldsymbol{u}_s = \left((u_s)_1, (u_s)_2, (u_s)_3, \ldots, (u_s)_m\right)^T \in \mathbb{R}^m$. Quantities defined at cells and at faces are related using an interpolator from cells to faces $\Gamma_{c\to s} \in \mathbb{R}^{m\times 3n}$:

$$\boldsymbol{u}_s \equiv \Gamma_{c\to s}\boldsymbol{u}_c. \tag{3}$$

The matrices $\Omega \in \mathbb{R}^{3n\times 3n}$, $\mathsf{C}\left(\boldsymbol{u}_s\right) \in \mathbb{R}^{3n\times 3n}$ and $\mathsf{D} \in \mathbb{R}^{3n\times 3n}$ are block diagonal matrices given by

$$\Omega = \mathsf{I}_3 \otimes \Omega_c, \qquad \mathsf{C}\left(\boldsymbol{u}_s\right) = \mathsf{I}_3 \otimes \mathsf{C}_c\left(\boldsymbol{u}_s\right), \qquad \mathsf{D} = \mathsf{I}_3 \otimes \mathsf{D}_c, \tag{4}$$

where $\mathsf{I}_3 \in \mathbb{R}^{3\times 3}$ is the identity matrix and $\Omega_c \in \mathbb{R}^{n\times n}$ is a diagonal matrix containing the cell-centered control volumes. $\mathsf{C}_c\left(\boldsymbol{u}_s\right) \in \mathbb{R}^{n\times n}$ and $\mathsf{D}_c \in \mathbb{R}^{n\times n}$ are the cell-centered convective and diffusive operators for a discrete scalar field, respectively. Finally, $\mathsf{G}_c \in \mathbb{R}^{3n\times n}$ is the discrete gradient operator, and the matrix $\mathsf{M} \in \mathbb{R}^{n\times m}$ is the face-to-center discrete divergence operator.

The 3−dimensional interpolator from cells to faces $\Gamma_{c\to s}$ is constructed as follows:

$$\Gamma_{c\to s} = N(\mathsf{I}_3 \otimes \Pi_{c\to s}), \tag{5}$$

where $\Pi_{c\to s} \in \mathbb{R}^{m\times n}$ is the scalar cell-to-face interpolator, and $N = (N_{s,x}N_{s,y}N_{s,z}) \in \mathbb{R}^{3m\times m}$, where $N_{s,i} \in \mathbb{R}^{m\times m}$ is a diagonal matrix containing the x_i spatial components of the face normal vectors.

Only five operators were needed to build this formulation: the cell-centered and staggered control volumes, Ω_c and Ω_s respectively, the face normal vectors N_s, the scalar cell-to-face interpolation operator $\Pi_{c\to s}$, and the cell-to-face divergence operator M. This simplicity not only eases the construction of the required operators but also improves the portability of a code developed within this framework, as demonstrated in [2]. The global kinetic energy $||\boldsymbol{u}_c||^2$ is conserved if [1]:

$$\mathsf{C}\left(\boldsymbol{u}_s\right) = -\mathsf{C}\left(\boldsymbol{u}_s\right)^T, \tag{6a}$$

$$-\left(\Omega\mathsf{G}_c\right)^T = \mathsf{M}\Gamma_{c\to s}, \tag{6b}$$

relating the gradient operator with the divergence operator and assuming the convective operator to be skew-symmetric. If $\mathsf{L}_c = \mathsf{M}_c\mathsf{G}_c$ is used to build the Poisson equation, the global kinetic energy is perfectly conserved. However, the well-known checkerboard problem is found with this approach. Alternatively, the use of $\mathsf{L} = \mathsf{MG}$ will be discussed in this work.

AN ENERGY-PRESERVING UNCONDITIONALLY STABLE PISO ALGORITHM

Assuming either explicit or implicit time integration, spatially discrete momentum equation (2a) can be rewritten as follows:

$$\mathbf{S}\boldsymbol{u}_c = \mathbf{r} - \mathsf{G}_c\boldsymbol{p}_c, \tag{7}$$

where $\mathbf{S} \in \mathbb{R}^{3n\times 3n}$ is the coefficient matrix after applying a discretization method (such as Finite Volume Method), and $\mathbf{r} \in \mathbb{R}^{3n\times 1}$ is a vector containing all the explicit terms apart from the pressure gradient. The coefficients are all known once the discretization procedure is selected.

By treating the pressure gradient as an explicit source and solving for the velocity, the momentum predictor is obtained:

$$\boldsymbol{u}_c^* = \mathbf{S}^{-1}\mathbf{r} - \mathbf{S}^{-1}\mathsf{G}_c\boldsymbol{p}_c^n. \tag{8}$$

Note that the momentum predictor $\boldsymbol{u}_c^*$ does not satisfy the continuity equation. To ensure so, a corrector step must be performed. The diagonal coefficients of $\mathbf{S}$ will be extracted in a diagonal matrix $\mathbf{A}$ (which will be easily invertible), and the off-diagonal coefficients will be keeped in a matrix $\mathbf{H}'$. Then, it is assumed that the diagonal matrix is acting on a new corrected velocity $\boldsymbol{u}_c^{**}$ while the off-diagonal part is acting on the predictor velocity $\boldsymbol{u}_c^*$.

The final algorithm reads [3]:

$$\begin{aligned} \boldsymbol{u}_c^* &= \mathbf{S}^{-1}\mathbf{r} - \mathbf{S}^{-1}\mathsf{G}_c\boldsymbol{p}_c, && \text{(9a)} \\ \mathsf{M}\tilde{\mathbf{A}}^{-1}\mathsf{G}\boldsymbol{p}_c^* &= \mathsf{M}\Gamma_{c\to s}\mathbf{A}^{-1}(\mathbf{r} - \mathbf{H}'\boldsymbol{u}_c^*) \longrightarrow \boldsymbol{p}_c^*, && \text{(9b)} \\ \boldsymbol{u}_c^{**} &= \mathbf{A}^{-1}(\mathbf{r} - \mathbf{H}'\boldsymbol{u}_c^*) - \mathbf{A}^{-1}\mathsf{G}_c\boldsymbol{p}_c^*, && \text{(9c)} \\ \boldsymbol{u}_s^{**} &= \Gamma_{c\to s}\mathbf{A}^{-1}(\mathbf{r} - \mathbf{H}'\boldsymbol{u}_c^*) - \tilde{\mathbf{A}}^{-1}\mathsf{G}\boldsymbol{p}_c^*. && \text{(9d)} \end{aligned}$$

Here, $\tilde{\mathbf{A}}^{-1} = diag(\Pi_{c\to s} vec(\mathbf{A}_c^{-1}))$ and and $\mathbf{A} = I_3 \otimes \mathbf{A}_c$ (the matrix $\mathbf{A}$ has three equal (diagonal) blocks). The PISO algorithm iterates through the corrector steps until the desired level of convergence is achieved. In this context, the utilization of the compact Laplacian operator L to mitigate checkerboard problems is assumed. Additionally, an (artificial) contribution to the kinetic energy will be introduced:

$$\boldsymbol{p}_c^T(\mathsf{M}\tilde{\mathbf{A}}^{-1}\mathsf{G} - \mathsf{M}_c\mathbf{A}^{-1}\mathsf{G}_c)\boldsymbol{p}_c = \boldsymbol{p}_c^T(\mathsf{L} - \mathsf{L}_c)\boldsymbol{p}_c. \qquad (10)$$

Ideally, the contribution of this term should be negative and maintained as small as possible to avoid introducing energy into our system, thereby preventing the simulation from destabilizing. At this point, note that each projection method that lead the equations to be solved in this way will encounter the same problem, for example, the classical Fractional Step Method [1], which is similar to the PISO algorithm but with no predictor step and corrector iterations.

GEOMETRICAL CONDITIONS OF THE PROJECTION METHOD IN ORDER TO BE UNCONDITIONALLY STABLE

Theorem.
Let us assume a general 2D or 3D mesh is constructed such that each control volume satisfies:

- **1.** $V_k = \sum_f \tilde{V}_{k,f} n_{i,f}^2, \forall k \in \{1, ..., n\}, i \in \{x, y, z\}$,
- **2.** $\sum_f \tilde{V}_{k,f} n_{i,f} n_{j,f} = 0, \forall k \in \{1, ..., n\}, i, j \in \{x, y, z\}, i \neq j$.

Then, $\mathsf{L} - \mathsf{L}_c$ is positive definite as long as the chosen interpolator is the volume weighted one (this is the unique possible choice of interpolator, for both $\mathsf{G}_c\boldsymbol{p}_c$ in the velocity correction equation and in the computation of $\tilde{\mathbf{A}}$). The converse is also true.

The volume weighted interpolator was introduced in [4], and can be constructed in any mesh as follows:

$$\Pi_{c\to s} = \Delta_s^{-1}\Delta_{sc}^T, \qquad (11)$$

where $\Delta_s \in \mathbb{R}^{m\times m}$ is a diagonal matrix containing the projected distances between two adjacent control volumes, and $\Delta_{sc} \in \mathbb{R}^{m\times n}$ is a matrix containing the projected distances between an adjacent cell node and its corresponding face. Fig.1 shows a representation of these distances.

Some consequences of the previous theorem are the following ones:

- Square and cubic meshes are stable when using the volume weighted interpolator even for highly distorted configurations.
- Triangular 2D meshes are stable when using the volume weighted interpolator and locating the cell-node at the circumcenter, even for highly distorted configurations, but tetrahedral meshes are not unconditionally stable.

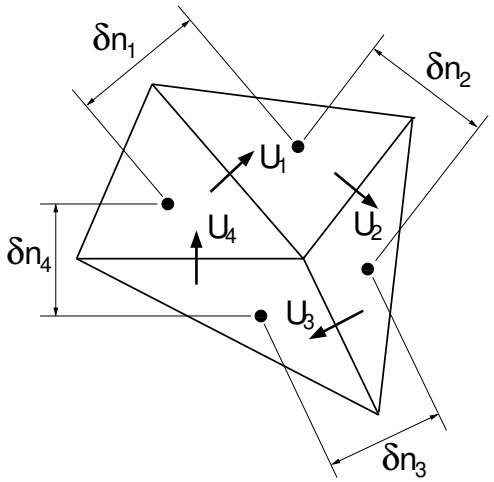

Figure 1: δn_i are the components of Δ_s, while the components of Δ_{sc} would be calculated in the same way but taking the distance between a control volume and their corresponding face centers.

CONCLUSIONS

Necessary and sufficient conditions in order to build an energy preserving unconditionally stable projection method on collocated unstructured grids are given in this work. Furthermore, different types of grids will be discussed in order to assess which are unconditionally stable even for highly distorted cases.

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