Mesh constraints for an energy preserving unconditionally stable projection method on collocated unstructured grids

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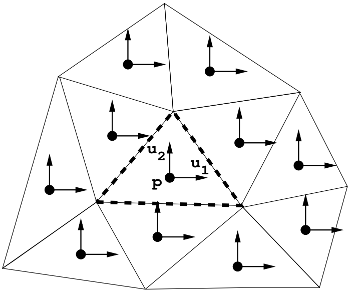
Motivation



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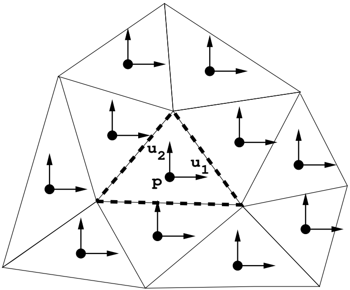

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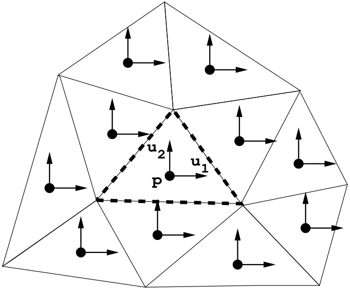

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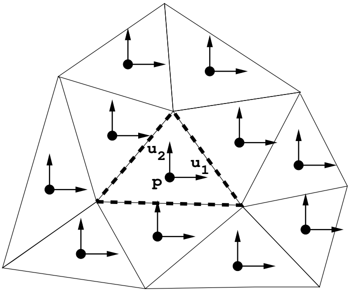

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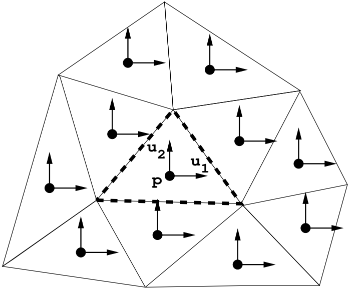

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- LES modelling → Framework to do DNS/LES modelling on complex geometries:
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  - Free of artificial numerical dissipation → Only dissipation from the LES model
  - Unconditionally stable
  - Easily portable (to other codes, platforms...)

1. Definition of basic collocated operators

Let us suppose we have n control volumes and m faces.

Finite volume discretization of incompressible NS equations on an arbitrary collocated mesh

$$\Omega \frac{d\mathbf{u}_c}{dt} + C(\mathbf{u}_s)\mathbf{u}_c = -D\mathbf{u}_c - \Omega G_c \mathbf{p}_c, \qquad (1)$$

$$M\mathbf{u}_s = \mathbf{0}_c. \qquad (2)$$

- $\mathbf{p}_c = (p_1, ..., p_n)^T \in \mathbb{R}^n$ is the cell-centered pressure.
- $\mathbf{u}_c = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^T \in \mathbb{R}^{3n}$, where $\mathbf{u}_i = ((u_i)_1, ..., (u_i)_n)^T$ are the vectors containing the velocity components corresponding to the x_i-spatial direction.
- $\mathbf{u}_s = ((u_s)_1, ..., (u_s)_m)^T \in \mathbb{R}^m$ is the staggered velocity.
- The velocities are related via the interpolator from cells to faces $\Gamma_{c\rightarrow s} \in \mathbb{R}^{m\times 3n} \implies \mathbf{u}_s = \Gamma_{c\rightarrow s}\mathbf{u}_c$.

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Definition of basic collocated operators

The (3D) interpolator from cells to faces can be constructed as follows:

$$\Gamma_{c \to s} = N\Pi, \tag{3}$$

where

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- $\Omega_c \in \mathbb{R}^{n\times n}$ is a diagonal matrix with the cell-centered volumes $\implies \Omega = I_3 \otimes \Omega_c$.
- $C_c(\mathbf{u}_s) \in \mathbb{R}^{n\times n}$ is the cell-centered convective operator for a discrete scalar field $\implies C(\mathbf{u}_s) = I_3 \otimes C_c(\mathbf{u}_s)$.
- $D_c \in \mathbb{R}^{n\times n}$ is the cell-centered diffusive operator for a discrete scalar field $\implies D = I_3 \otimes D_c$.

Finally,

- $G_c \in \mathbb{R}^{3n\times n}$ represents the discrete collocated gradient.
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Other useful operators

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\begin{aligned}
\boxed{G = -\Omega_s^{-1} M^T,} \\
L = MG = -M\Omega_s^{-1} M^T, \\
L_c = M_c G_c = -M\Gamma_{c\to s}\Omega^{-1}\Gamma_{c\to s}^T M^T, \\
\Gamma_{s\to c} = \Omega^{-1}\Gamma_{c\to s}^T \Omega_s.
\end{aligned} \tag{4}
$$

where G is the center-to-face staggered gradient, L is the Laplacian operator, L_c is the collocated-Laplacian operator and $\Gamma_{s\to c}$ is the face-to-cell interpolator.

For more information about Symmetry-Preserving discretization consult: *F.X. Trias, O. Lehmkuhl, A. Oliva, C.D. Perez-Segarra, and R.W.C.P. Verstappen. Symmetry-preserving discretization of Navier-Stokes equations on collocated unstructured meshes. Journal of Computational Physics, 258:246–267, 2014.*

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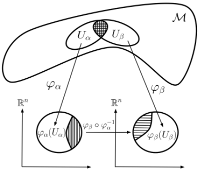
2. Conservation of global kinetic energy

Global kinetic energy equation

$$\frac{d||\mathbf{u}_c||^2}{dt} = -\mathbf{u}_c^T(C(\mathbf{u}_s) + C^T(\mathbf{u}_s))\mathbf{u}_c - \mathbf{u}_c^T(D + D^T)\mathbf{u}_c - \mathbf{u}_c^T\Omega G_c\mathbf{p}_c - \mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c. \quad (5)$$

In the absence of diffusion, that is, $D = 0$, the global kinetic energy is conserved if:

- $C(\mathbf{u}_s) = -C^T(\mathbf{u}_s)$, i.e, the convective operator should be skew-symmetric.
- $(-\Omega G_c)^T = M\Gamma_{c\rightarrow s}$, because $M\mathbf{u}_s = \mathbf{0}_c$.



Maths

Differential geometry: k-forms $\Lambda^k(T_p^*\mathcal{M})$

(Continuous) symmetry-preserving

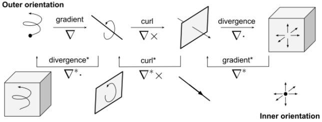
Algebraic topology: k-chains

Physics

Conservation of energy

Imposed by physics!

Discrete symmetry-preserving



$$\Lambda^k(\Omega) \xrightarrow{\pi_h} \Lambda_h^k(\Omega, C_k), \quad \Lambda^k(\Omega) \xrightarrow{\mathcal{R}} C^k(D), \quad C^k(D) \xrightarrow{\mathcal{I}} \Lambda_h^k(\Omega, C_k)$$

Conservation of global kinetic energy

Global kinetic energy equation with skew-symmetric convective operator

$$\frac{d||\mathbf{u}_c||^2}{dt} = -\mathbf{u}_c^T(D + D^T)\mathbf{u}_c - \mathbf{u}_c^T \Omega G_c \mathbf{p}_c - \mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c.$$

In absence of diffusion, that is $D = 0$, the global kinetic energy is conserved if:

- $(-\Omega G_c)^T = M\Gamma_{c\rightarrow s}$, because $M\mathbf{u}_s = \mathbf{0}_c$ (But this relation is exact ONLY in staggered configurations!).

In collocated framework, we either solve:

$$M\mathbf{u}_s = 0 \rightarrow Lp_c = M\Gamma_{c\rightarrow s}\mathbf{u}_c^p \rightarrow \text{Kinetic Energy Error} \quad (6)$$

$$M_c\mathbf{u}_c = 0 \rightarrow L_c p_c = M\Gamma_{c\rightarrow s}\mathbf{u}_c^p \rightarrow \text{Checkerboard} \quad (7)$$

Conservation of global kinetic energy

Global kinetic energy equation with skew-symmetric convective operator

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In collocated framework and explicit time integration, the (artificial) kinetic energy added is given by:

$$-\mathbf{p}_c^T G_c^T\Omega^T\mathbf{u}_c = \mathbf{p}_c^T(L - L_c)\mathbf{p}_c\Delta t \quad (8)$$

A stable pressure gradient interpolation

- The volume-weighted interpolator can be constructed as:

$$\Pi_{c \to s} = \Delta_s^{-1} \Delta_{sc}^T, \qquad (9)$$

where $\Delta_s \in \mathbb{R}^{m \times m}$ is a diagonal matrix containing the projected distances between two adjacent control volumes, and $\Delta_{sc} \in \mathbb{R}^{m \times n}$ contains the projected distances between an adjacent cell node and its corresponding face.

Volume-weighted interpolation: $\phi_f = \frac{\tilde{V}_{1,f}}{\tilde{V}_{1,f} + \tilde{V}_{2,f}} \phi_{c1} + \frac{\tilde{V}_{2,f}}{\tilde{V}_{1,f} + \tilde{V}_{2,f}} \phi_{c2}$.

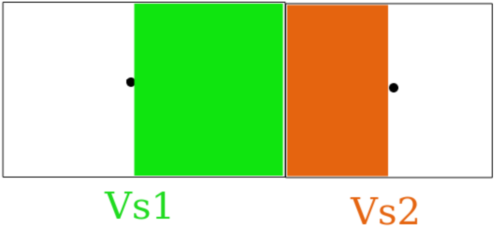

Figure 2: Volume-weighted volumes

Conservation of global kinetic energy

Stable solutions → Eigenvalues of $L - L_c$ negative.

Theorem

Assumptions:

Conservation of global kinetic energy

Stable solutions $\rightarrow$ Eigenvalues of $L - L_c$ negative.

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Assumptions:

- Our projection method adds a kinetic energy error of the form $\mathbf{p}_c^T(L - L_c)\mathbf{p}_c^T$ (such as the FSM or PISO).

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1. The volume-weighted cell-to-face interpolator is used for the pressure gradient interpolator.
2. $$V_k = \sum_{f \in F(k)} \tilde{V}_{k,f} n_{i,f}^2, \quad \forall k \in \{1, ..., n\}, \quad i \in \{x, y, z\} \tag{10}$$

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$$\sum_{f \in F(k)} \tilde{V}_{k,f} n_{i,f} n_{j,f} \leq 0, \quad \forall k \in \{1, ..., n\}, \quad i, j \in \{x, y, z\}, \quad i \neq j, \tag{11}$$

Conservation of global kinetic energy

What is the theorem saying:

- Under these assumptions, the method is always unconditionally stable.
- The volume-weighted interpolator is strictly needed for the result.
- The theorem holds for both explicit and implicit time integration.

What is the theorem not saying:

- These are the unique assumptions in order to have stable simulations.

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Cartesian hexahedral meshes always give stable results when using the volume-weighted interpolator.

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Corollary 1

Cartesian hexahedral meshes always give stable results when using the volume-weighted interpolator.

Corollary 2

Triangular meshes give stable results when using the volume-weighted interpolator if the node is placed at the circumcenter.

Numerical robustness

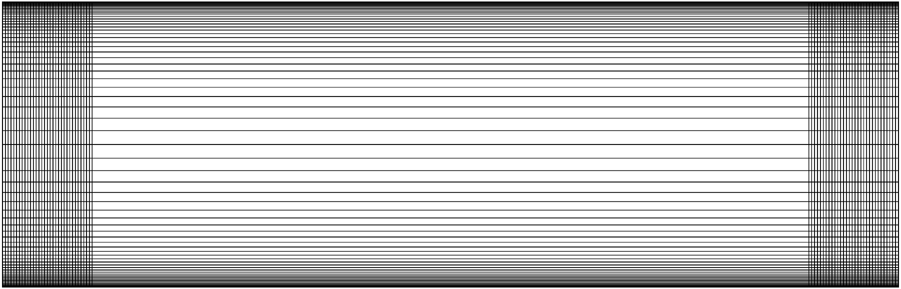

Figure 3: Highly distorted mesh used to test the method's robustness in a $Re_\tau = 395$ channel flow. Maximum aspect ratio is 250.

Numerical robustness

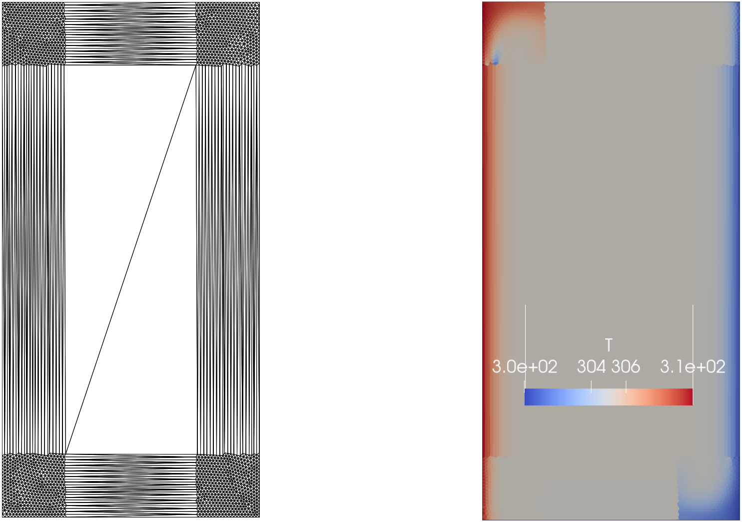

Figure 4: Test of the method's robustness in a $Ra = 10^6$ differentially heated cavity.

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Ongoing work

- Find the conditions for tetrahedral meshes in order to satisfy the geometrical conditions of the theorem.