Beyond classical stability analysis on Runge-Kutta schemes: positivity and phase preservation

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Introduction

Runge-Kutta integration in Navier-Stokes

Performed according to Sanderse and Koren [1]:

$$\mathrm{u}_i^* = \mathrm{u}_n + \Delta t \sum_{j=1}^{i-1} a_{ij} \mathrm{F}_j$$

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with self-adaptive timestep, adapted from Trias and Lehmkuhl[2] to Runge-Kutta integration.

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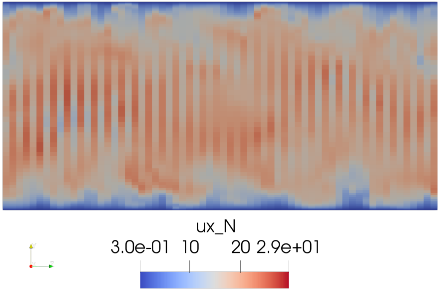

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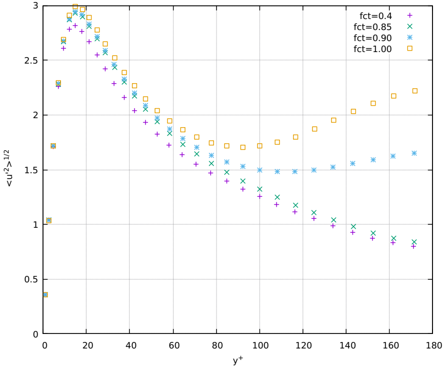

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Positivity preservation
Real eigenvalue

- Consider the following ODE,

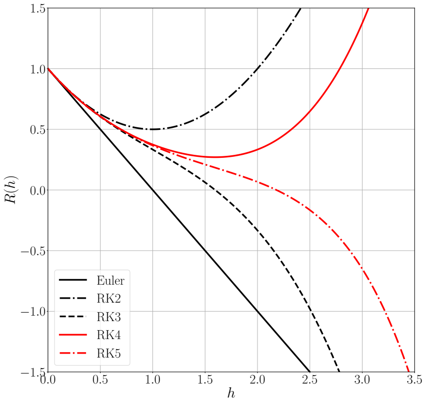
$$\frac{d\phi}{dt} = \lambda\phi,$$

which solution is known as $\phi(t) = \phi_0 e^{\lambda t}$, where $\phi_0 = \phi(0)$.

- If $\lambda = -1$...

$$R(h) = 1 + \sum_{k=1}^{s} \frac{(-1)^k}{k!} h^k,$$

so that $\phi^{n+1} \approx R(h)\phi^n$.



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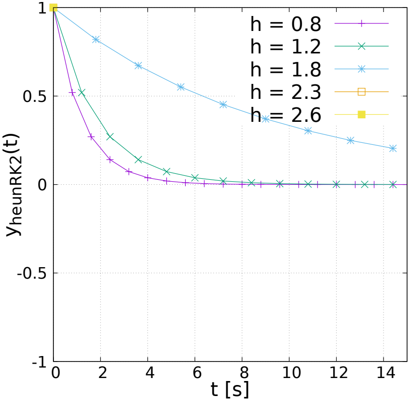

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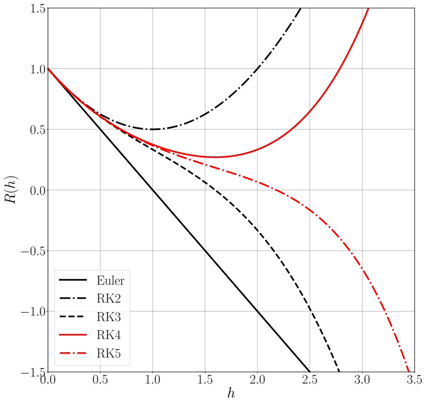

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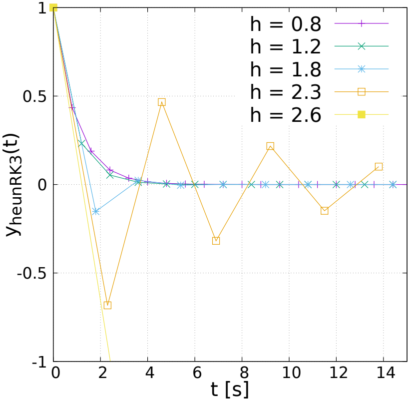


Positivity preservation

Complex eigenvalue

Let $\lambda = -||\lambda||e^{-i\varphi}$...

The stability polynomial can be rewritten as,

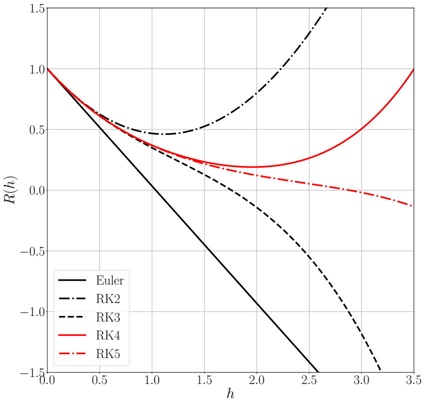
$$R(h\lambda) = 1 + \sum_{k=1}^{s} \frac{(-1)^k}{k!}(h||\lambda||)^k \cos(k\varphi) + i\sum_{k=1}^{s} \frac{(-1)^{k+1}}{k!}(h||\lambda||)^k \sin(k\varphi) = R_r + iR_i$$

- Considering $\phi^n = \phi_r^n + i\phi_i^n$...

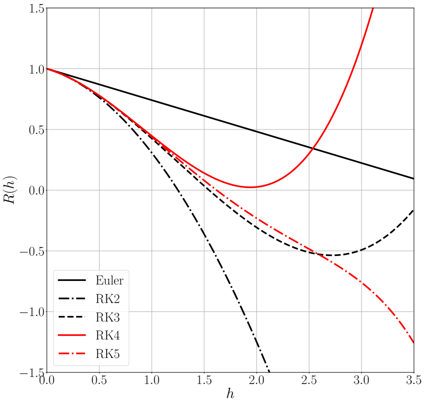
$$\begin{pmatrix} \phi_r^{n+1} \\ \phi_i^{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} R_r & -R_i \\ R_i & R_r \end{pmatrix}}_{A} \begin{pmatrix} \phi_r^n \\ \phi_i^n \end{pmatrix}$$

- It will preserve positivity if $x^T A x > 0$, for an arbitrary $x \in \mathbb{R}^2$, which is achieved if $R_r > 0$.

Positivity preservation
Complex eigenvalue



$\varphi = 15^o$



$\varphi = 75^o$

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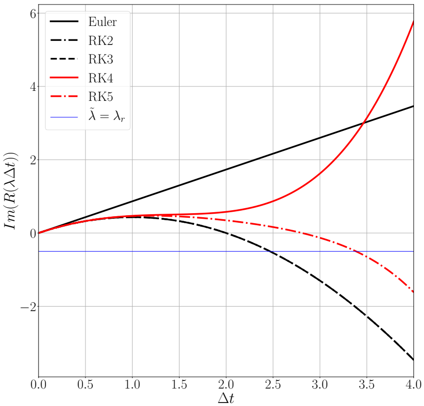
Positivity-preserving solutions are obtained when...

$$R_r > 0$$

Phase preservation

- So... what if $R_i < 0$? For $\varphi = 60^o$...



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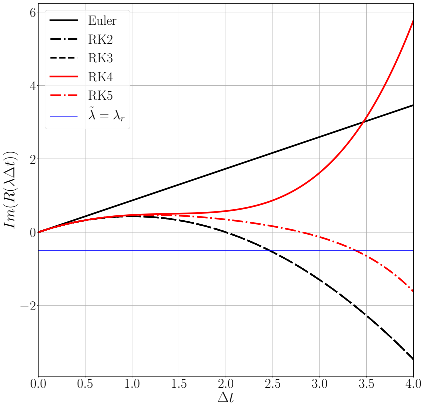

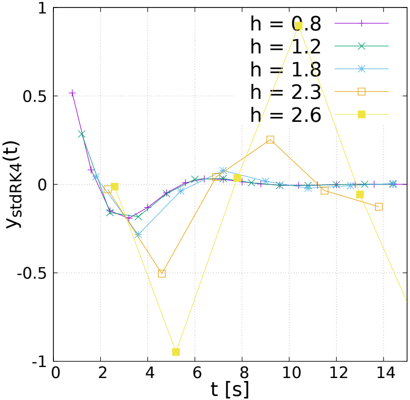

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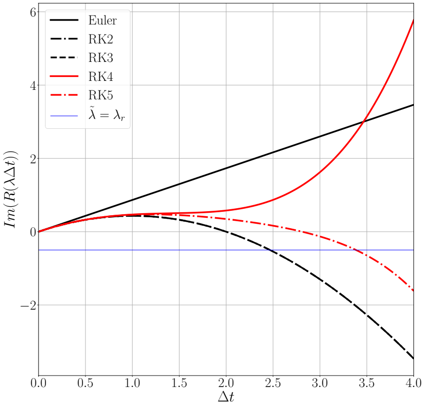

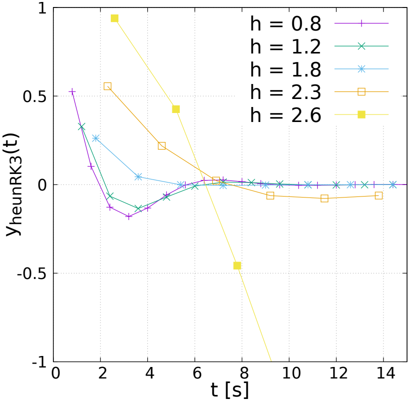

Phase preservation

- The phase of the solution changes for a big enough step size h.
  - As for the positivity preservation, depends on RK scheme as well as the ODE eigenvalues.

Phase-preserving solutions are obtained when...

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Revisiting the stability region

Original standpoint

Linear stability analysis

$$\phi^{n+1} \approx R(\lambda h)\phi^n$$

- For explicit Runge-Kutta schemes such that $s = p$:

$$R(\lambda h) = 1 + \sum_{k=1}^{p} \frac{(\lambda h)^k}{k!}$$

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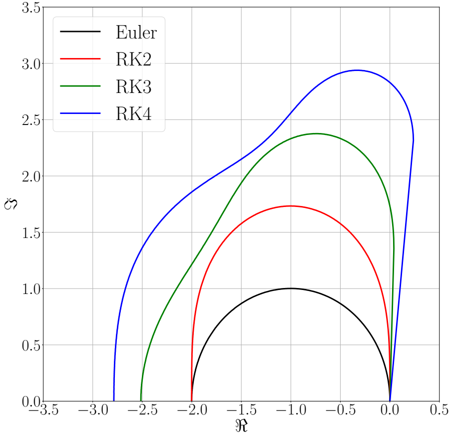

Revisiting the stability region

Additional conditions

Application of the previous results...

- Numerical stability

- Required conditions:

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  - Non-physical oscillations disappear

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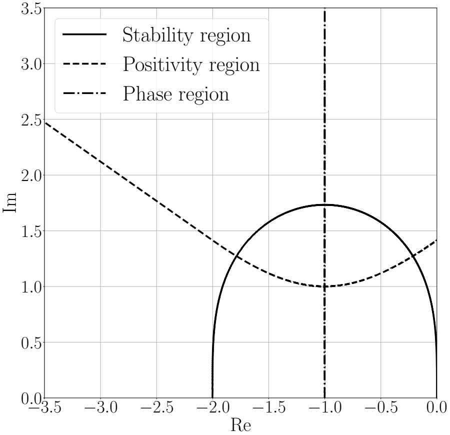

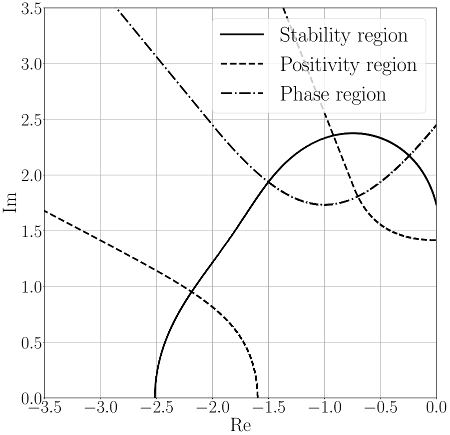

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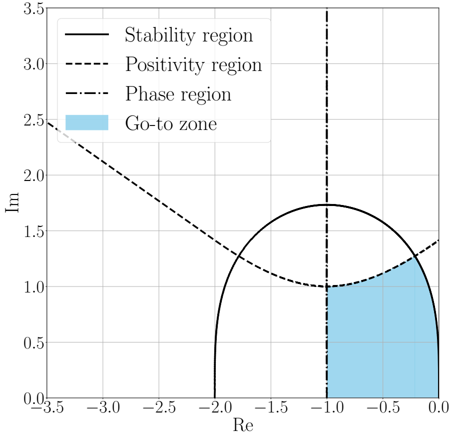
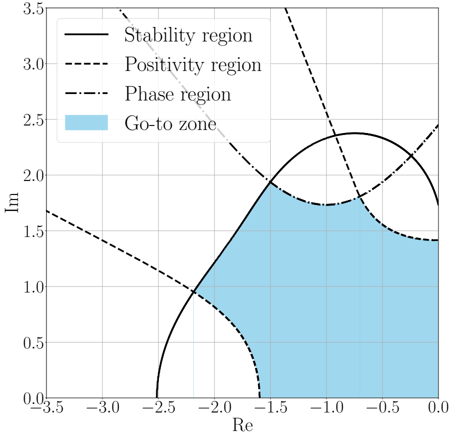
Go-to zone





Concluding remarks

- Non-physical oscillations were observed in a turbulent channel flow simulation
- Effect of a large stable h in the numerical solution of the ODE
  - Can generate those non-physical oscillations
  - Can generate changes on the phase of the solution $\rightarrow$ sort of dispersion error?
- Found to be due to the sign of the stability polynomial
- Representation of the positivity region and phase region
- Determination of the go-to zone