A general method to compute numerical dispersion error

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  - Except if Spectral Methods are used, where derivative is imposed to be exact: $f'(k) = kf(k)$.
- How is Numerical Dispersion usually studied? By means of a **Fourier Transform.**

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- A methodology that allows studying dispersion in a general mesh would be interesting.
  - Numerical dispersion is, then, a function of the studied mesh.
- Instead of using the sinusoids base, use an orthogonal base extracted from studied mesh.
  - For example, **eigenvectors of the discrete Laplacian matrix.**

Methodology: Calculus background (I)

Let $\Phi = \{\phi_{-N}(x), \phi_{-N+1}(x), \ldots \phi_{-1}(x), \phi_0(x), \phi_1(x), \ldots \phi_N(x)\}$ be an orthonormal basis of functions in a domain Ω_x.

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$T : \mathcal{L}^2(\Omega_x, x) \mapsto \mathbb{C}^{2N+1}$; $T : f(x) \mapsto (\alpha_m) \in \mathbb{C}^{2N+1}$, where

$$\alpha_m = \langle f \mid \phi_m \rangle_{\Omega_x} = \int_{\Omega_x} f(x)\overline{\phi_m}(x)\, dx, \quad (1)$$

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$T^{-1} : \mathbb{C}^{2N+1} \mapsto \mathcal{L}^2(\Omega_x, x)$, $T : (\alpha_m) \in \mathbb{C}^{2N+1} \mapsto f(x)$.

$$f(x) \simeq S_N = \sum_{m=-N}^{N} \alpha_m \phi_m(x); \quad \lim_{N\to\infty} S_N = f(x). \tag{2}$$

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Where its elements $(\Gamma)_{mn} = \gamma_{mn} = \langle \phi'_m \,|\, \phi_n \rangle_{\Omega_x}$. The structure of Γ will provide information about the errors produced during differentiation.

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$$\widetilde{\gamma_{mn}} = \langle A\phi_m \,|\, \phi_n \rangle \tag{5}$$

$$Im(\widetilde{\gamma_{mn}}) = \langle C\phi_m \,|\, \phi_n \rangle = \frac{\langle A\phi_m \,|\, \phi_n \rangle - \langle \phi_m \,|\, A\phi_n \rangle}{2} \tag{6}$$

$$Re(\widetilde{\gamma_{mn}}) = \langle D\phi_m \,|\, \phi_n \rangle = \frac{\langle A\phi_m \,|\, \phi_n \rangle + \langle \phi_m \,|\, A\phi_n \rangle}{2} \tag{7}$$

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- Retain the concept of mesh connectivity without being restrained to mesh uniformity.

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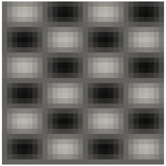

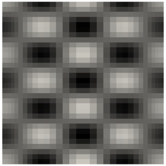

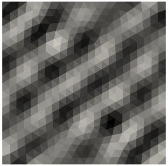

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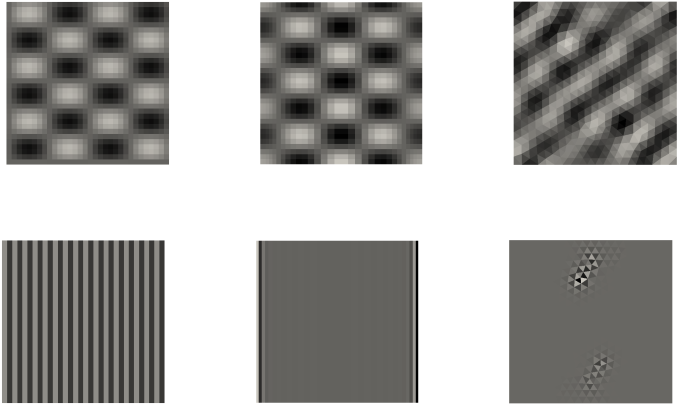

Methodology: Phase

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Sinusoids orthonormal basis have a free parameter: the phase of the function.

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This allows to obtain the average of the recovered numerical eigenvalue.

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The matrix containing eigenvectors is multiplied by a rotation matrix with a random phase.

- And this is repeated N times (5000) to ensure a correct average.

Test cases: Selected cases

Used schemes

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Mixture of linear and non-linear schemes:

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Mixture of linear and non-linear schemes:

- Symmetry preserving of 2^{nd} and 6^{th} order {SP2, SP6} Dispersion relation preserving of 4^{th} and 6^{th} order {DRP4, DRP6}, and Moving Least squares of 6^{th} order {MLS3}

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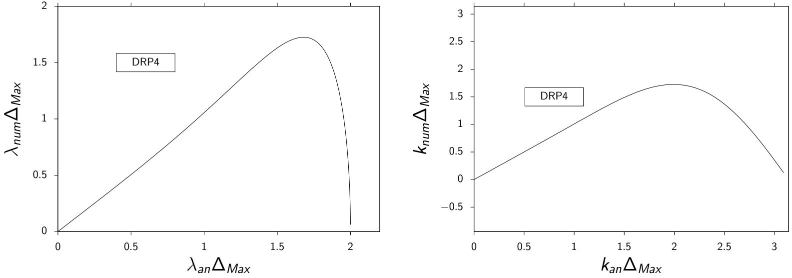
Using 30 one-dimensional stretched meshes:

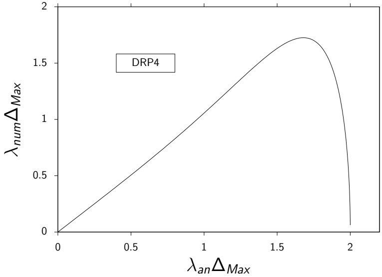
- From 0 to 5% stretching ratio
- Δx_{min} from $1/32$ to $1/512$.

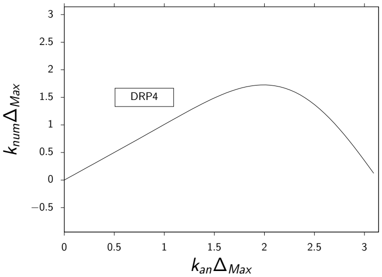
Test cases: Results



Left: Numerical eigenvalues. Right: Numerical wavenumbers. Uniform mesh.

Test cases: Results





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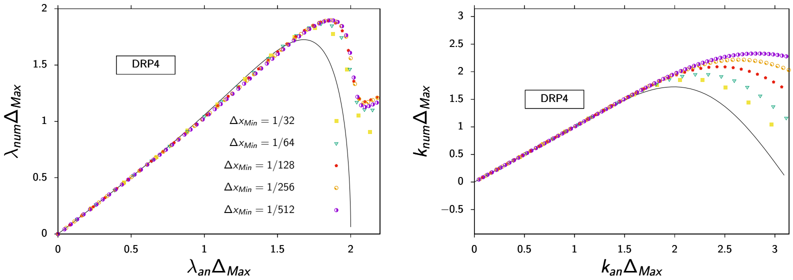
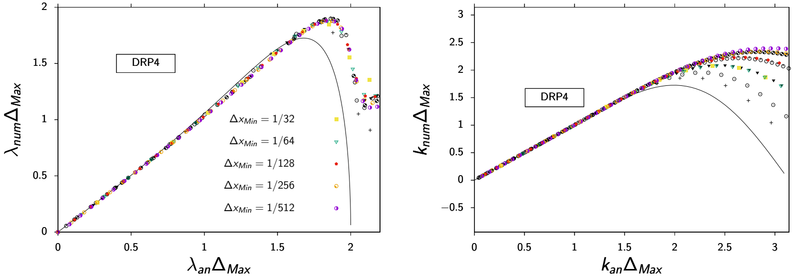
$$\lambda_{an} = \frac{4}{\Delta x} sin^2 \left(k_{an} \Delta x \right)$$

Test cases: Results



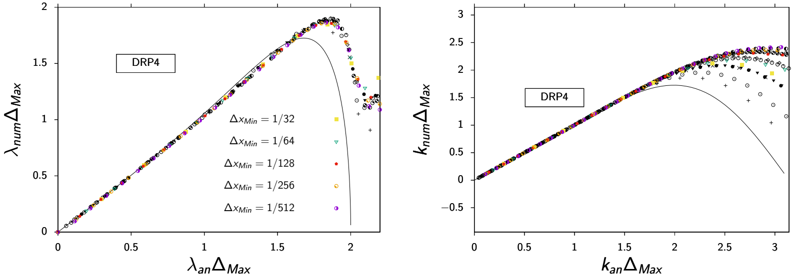
Left: Numerical eigenvalues. Right: Numerical wavenumbers.
Until 1% stretching.

Test cases: Results



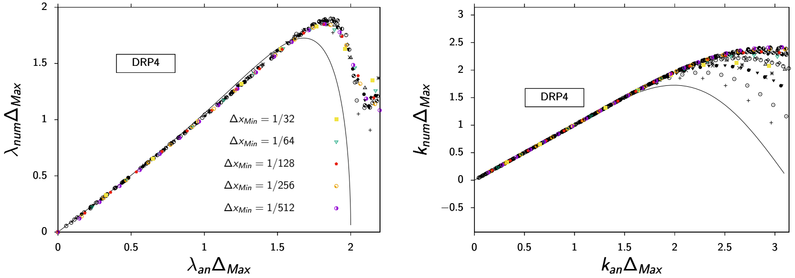
Left: Numerical eigenvalues. Right: Numerical wavenumbers.
Until 2% stretching.

Test cases: Results



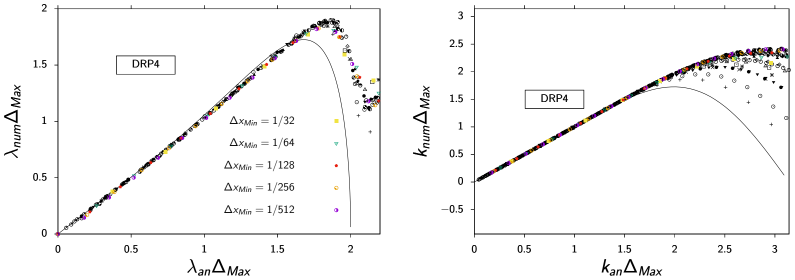
Left: Numerical eigenvalues. Right: Numerical wavenumbers.
Until 3% stretching.

Test cases: Results



Left: Numerical eigenvalues. Right: Numerical wavenumbers.
Until 4% stretching.

Test cases: Results



Left: Numerical eigenvalues. Right: Numerical wavenumbers.
Until 5% stretching.

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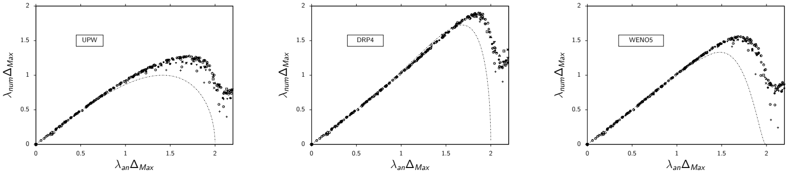

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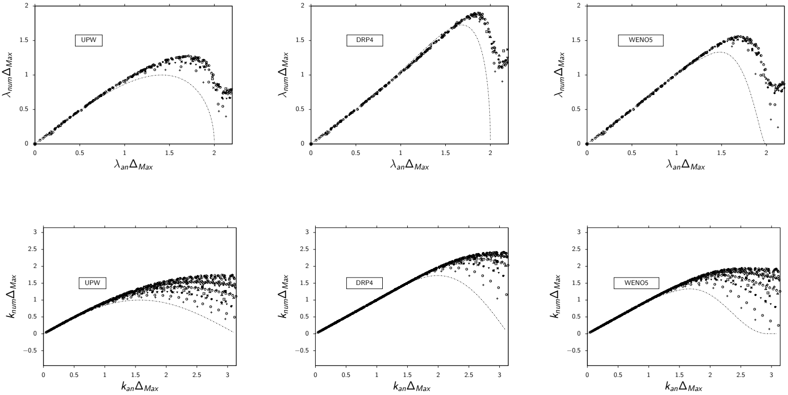
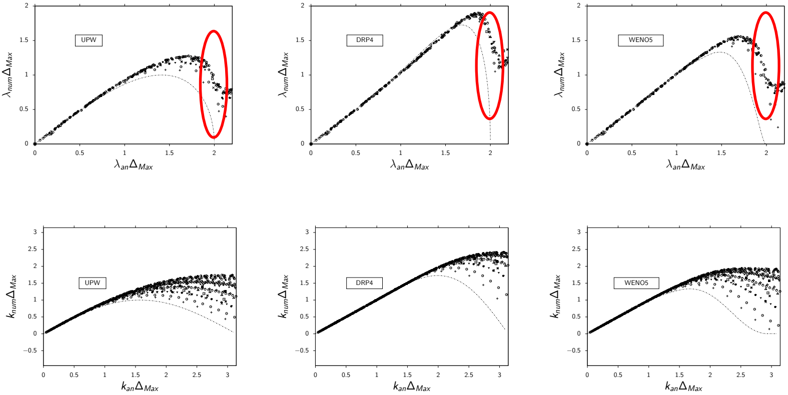


Test cases: Results

Always at $\lambda \Delta x_{Max} = 2$



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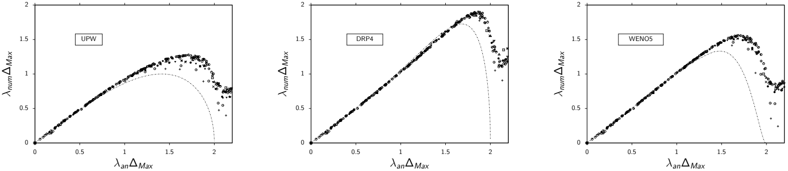
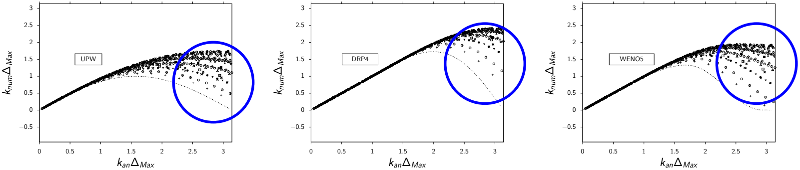


Not a clear cut-off k



Test cases: Results

Stretch. [%]	*SP2*	*DRP4*	*DRP6*	*SP6*	*MLS3*
0	1	1.7254	1.8368	1.586	1.5615
1	1.203	1.8884	1.9638	1.7688	1.7466
2	1.2396	1.8792	1.9466	1.7704	1.7488
3	1.2501	1.856	1.9239	1.7564	1.7369
4	1.2512	1.8364	1.9018	1.7412	1.7205
5	1.2432	1.8223	1.8748	1.7268	1.708
AVG	1.2374	1.8565	1.9222	1.7527	1.7322

Table: Non-dimensional maximum eigenvalue normalized respect maximum eigenvalue for second-order symmetry preserving in uniform meshes, linear schemes.

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4	1.2512	1.8364	1.9018	1.7412	1.7205
5	1.2432	1.8223	1.8748	1.7268	1.708
AVG	1.2374	1.8565	1.9222	1.7527	1.7322

Table: Non-dimensional maximum eigenvalue normalized respect maximum eigenvalue for second-order symmetry preserving in uniform meshes, linear schemes.

Test cases: Results

Stretch. [%]	*SP*2	*DRP*4	*DRP*6	*SP*6	*MLS*3
0	1	1.7254	1.8368	1.586	1.5615
1	1.203	1.8884	1.9638	1.7688	1.7466
2	1.2396	1.8792	1.9466	1.7704	1.7488
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Stretch. [%]	*WENO3*	*WENO5*	*WENO7*	*MM*	*SB*	*VL*
0	1.1667	1.3317	1.4529	1.2526	1.4053	1.3237
1	1.3747	1.52	1.6315	1.4713	1.6272	1.543
2	1.4036	1.5402	1.6432	1.4946	1.6444	1.566
3	1.4072	1.5362	1.6377	1.4966	1.637	1.5611
4	1.4047	1.5296	1.6249	1.4973	1.6344	1.5573
5	1.3889	1.5183	1.6132	1.4736	1.6205	1.5374
AVG	1.3958	1.5289	1.6301	1.4867	1.6327	1.553

Table: Non-dimensional maximum eigenvalue normalized respect maximum eigenvalue for first-order upwind in uniform meshes, non-linear schemes.

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Test cases: Results

Stretch. [%]	*WENO*3	*WENO*5	*WENO*7	*MM*	*SB*	*VL*
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Test cases: Computational cost

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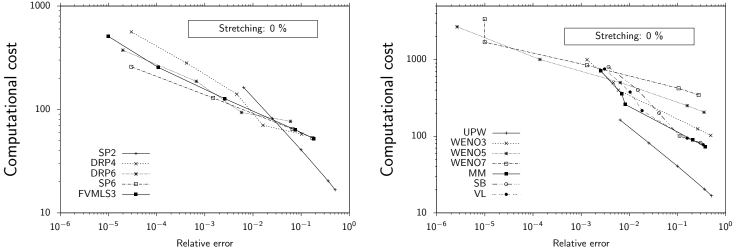
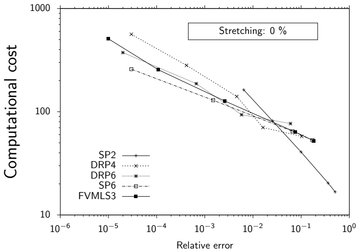
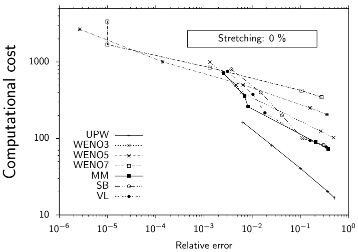


Computational cost vs relative error at uniform meshing.

Test cases: Computational cost





Computational cost vs relative error at uniform meshing.

At uniform meshing...

High-order schemes are more cost-effective. They achieve lesser relative errors than low-order schemes for the same computational cost.

Test cases: Computational cost

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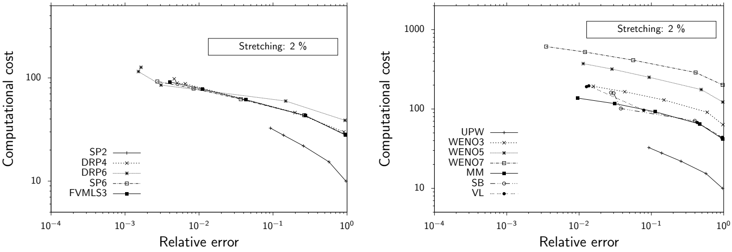
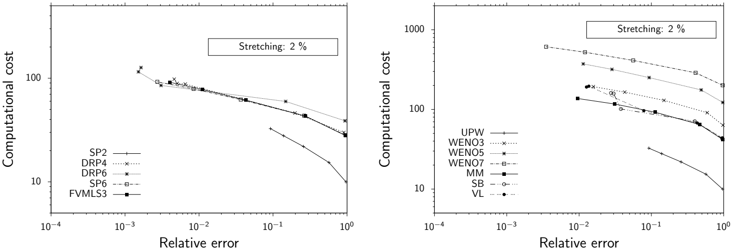


Computational cost vs relative error at 2% stretching.

Test cases: Computational cost



Computational cost vs relative error at 2% stretching.

At slightly stretched...

All schemes present a higher relative error: High-order lose two order of magnitude; low-order just one. High-order schemes seem to have lost order of accuracy. For errors in range, low-order are more cost effective.

Test cases: Computational cost

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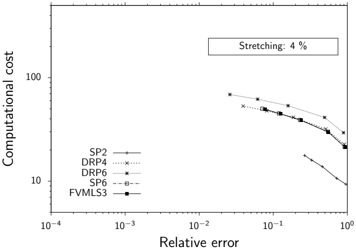
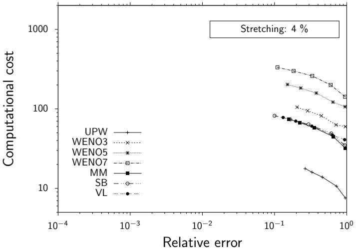
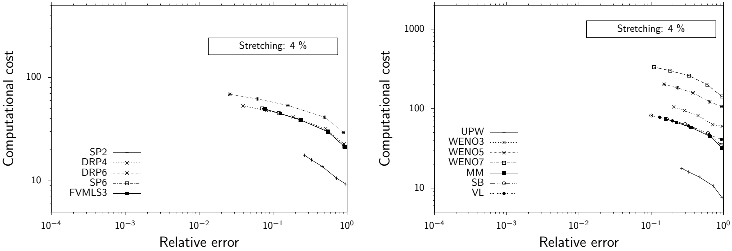




Computational cost vs relative error at 4% stretching

Test cases: Computational cost



Computational cost vs relative error at 4% stretching

At highly stretched...

All schemes relative error is higher than 1%. Non-linear schemes do not behave correctly.

Conclusions and further work

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A methodology to compute dispersion error in a general framework has been developed.

- No mesh uniformity nor periodic boundary conditions are required. Instead, uses the eigenvectors of the **discrete Laplacian operator**.

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- No mesh uniformity nor periodic boundary conditions are required. Instead, uses the eigenvectors of the **discrete Laplacian operator**.

A new numerical relation between expected and recovered eigenvalues has been found for studied schemes.

- Stretched meshes, independently on the stretching factor used on the study range, colapse onto the **same plot**, which is not the same that if uniform meshes are used.

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Maximum allowed eigenvalue with minimal dispersion directly related to maximum mesh size ($\lambda \Delta x_{Max} < 2$).

- A maximum allowed frequency related to mesh size does not appear. Instead, results are mesh dependent.

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Further work

Propose a meshing technique leading to dispersion reduction.

- Select the **most appropiate scheme for a given mesh**.

Thanks for your attention