On the interpolation problem for the Poisson equation on collocated meshes.

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Index

1. Motivation: unphysical velocities appearance in highly distorted meshes.
2. Symmetry-Preserving discretization of NS equations on collocated unstructured grids.
3. Theoretical results for Cartesian meshes.
4. Numerical results for unstructured meshes.
5. Conclusions.

Unphysical velocities appearance in highly distorted meshes

Motivation: Appearance of unphysical velocities in highly distorted meshes.

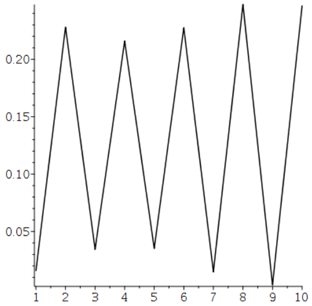

Figure 1: Vel. for max. aspect ratio 1 (mid-point scheme)

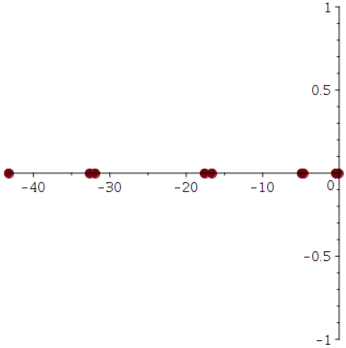

Figure 2: Eigenvalues of $L - L_c$

Unphysical velocities appearance in highly distorted meshes

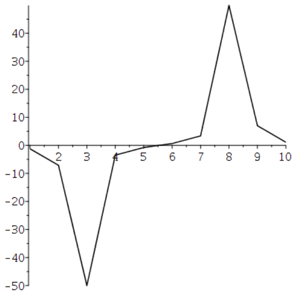

Figure 3: Vel. for max. aspect ratio 20 (mid-point scheme)

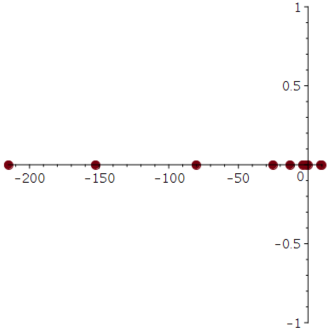

Figure 4: Eigenvalues of $L - L_c$

Unphysical velocities appearance in highly distorted meshes

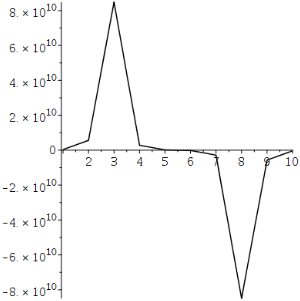

Figure 5: Vel. for max. aspect ratio 40 (mid-point scheme)

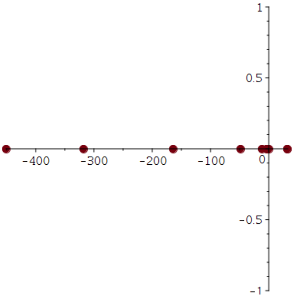

Figure 6: Eigenvalues of $L - L_c$

Unphysical velocities appearance in highly distorted meshes

- **Mid-point scheme:** $u_f = \frac{1}{2}(u_{c1} + u_{c2})$.
- **Volume weighted scheme:** $u_f = \frac{V_{s1}}{V_{s1}+V_{s2}} u_{c1} + \frac{V_{s2}}{V_{s1}+V_{s2}} u_{c2}$.

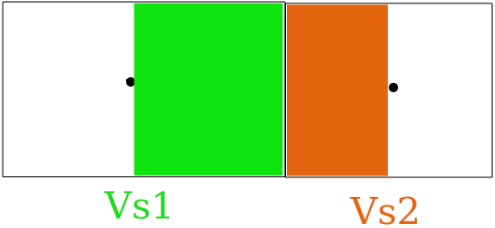

Figure 7: Volume weighted volumes

Unphysical velocities appearance in highly distorted meshes

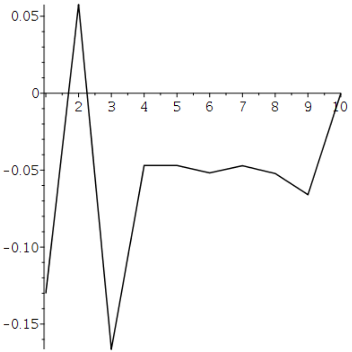

Figure 8: Vel. for max. aspect ratio 40 (volume weighted scheme)

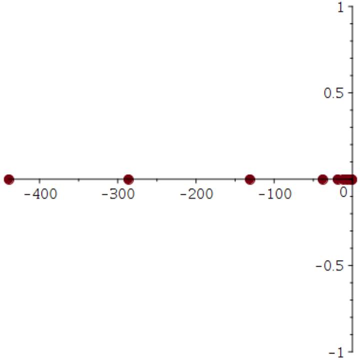

Figure 9: Eigenvalues of $L - L_c$

Definition of basic collocated operators

Incompressible NS equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{Re}\Delta \mathbf{u} - \nabla p, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2)$$

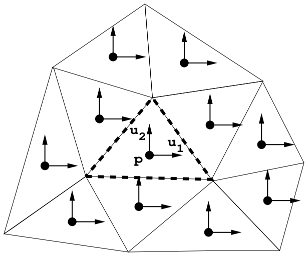

Figure 10: General unstrucured mesh.

Definition of basic collocated operators

Let us suppose we have n control volumes and m faces.

Finite volume discretization of incompressible NS equations on an arbitrary collocated mesh

$$\Omega \frac{d\mathbf{u}_c}{dt} + C(\mathbf{u}_s)\mathbf{u}_c = -D\mathbf{u}_c - \Omega G_c \mathbf{p}_c, \tag{3}$$

$$M\mathbf{u}_s = \mathbf{0}_c. \tag{4}$$

- $\mathbf{p}_c = (p_1, ..., p_n)^* \in \mathbb{R}^n$ is the cell-centered pressure.
- $\mathbf{u}_c = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^* \in \mathbb{R}^{3n}$, where $\mathbf{u}_i = ((u_i)_1, ..., (u_i)_n)^*$ are the vectors containing the velocity components corresponding to the x_i-spatial direction.
- $\mathbf{u}_s = ((u_s)_1, ..., (u_s)_m)^* \in \mathbb{R}^m$ is the staggered velocity.
- The velocities are related via the interpolator from cells to faces $\Gamma_{c\rightarrow s} \in \mathbb{R}^{m\times 3n} \implies \mathbf{u}_s = \Gamma_{c\rightarrow s}\mathbf{u}_c$.

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Definition of basic collocated operators

- $\boxed{\Omega_c \in \mathbb{R}^{n\times n}}$ is a diagonal matrix with the cell-centered volumes $\implies \Omega = I_3 \otimes \Omega_c$.
- $\boxed{C_c(\mathbf{u}_s) \in \mathbb{R}^{n\times n}}$ is the cell-centered convective operator for a discrete scalar field $\implies C(\mathbf{u}_s) = I_3 \otimes C_c(\mathbf{u}_s)$.
- $\boxed{D_c \in \mathbb{R}^{n\times n}}$ is the cell-centered diffusive operator for a discrete scalar field $\implies D = I_3 \otimes D_c$.

Finally,

- $G_c \in \mathbb{R}^{3n\times n}$ represents the discrete collocated gradient.
- $M \in \mathbb{R}^{n\times m}$ is the face-to-cell discrete divergence operator.

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Other useful operators

$$
\begin{aligned}
\boxed{G = -\Omega_s^{-1} M^*,} \\
L = MG = -M\Omega_s^{-1} M^*, \\
L_c = M_c G_c = -M\Gamma_{c\to s}\Omega^{-1}\Gamma_{c\to s}^* M^*, \\
\Gamma_{s\to c} = \Omega^{-1}\Gamma_{c\to s}^*\Omega_s.
\end{aligned} \tag{5}
$$

where G is the center-to-face staggered gradient, L is the Laplacian operator, L_c is the collocated-Laplacian operator and $\Gamma_{s\to c}$ is the face-to-cell interpolator.

For more information about Symmetry-Preserving discretization consult: *F.X. Trias, O. Lehmkuhl, A. Oliva, C.D. Perez-Segarra, and R.W.C.P. Verstappen. Symmetry-preserving discretization of Navier-Stokes equations on collocated unstructured meshes. Journal of Computational Physics, 258:246–267, 2014.*

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FSM iterative Poisson equation in collocated meshes

$$L\tilde{p}_c^{n+1} = M_c u_c^n \quad \longrightarrow \quad u_c^{n+1} = u_c^n - G_c \tilde{p}_c^{n+1}, \tag{6}$$

where $M_c = M\Gamma_{c\to s}$and $G_c = \Gamma_{s\to c} G = -\Omega_c^{-1}\Gamma_{c\to s}^T M^T$ are the collocated divergence and the collocated gradient. Developing the correction in u_c^n:

$$u_c^n = u_c^{n-1} - G_c\tilde{p}_c^n = u_c^{n-2} - G_c\tilde{p}_c^n - G_c\tilde{p}_c^{n-1} = \cdots = u_c^p - G_c \sum_{i=1}^{n} \tilde{p}_c^i \tag{7}$$

So, the acumulated pressure at n iteration is:

$$p_c^n = \sum_{i=1}^{n} \tilde{p}_c^i \tag{8}$$

Introducing all this in (7) we obtain:

$$Lp_c^{n+1} = M\Gamma_{c\to s} u_c^p + (L - L_c)p_c^n. \tag{9}$$

Theoretical results for Cartesian meshes

We will understand a Cartesian mesh as a mesh whose faces are parallel to Cartesian axis. For our purpose, we can have elements with different volumes.

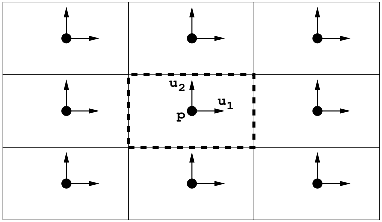

Figure 11: Cartesian mesh

Theoretical results for Cartesian meshes

Result 1

Suppose

$$Lp^{n+1} = b + (L - L_c)p^n \tag{10}$$

is an iterative equation with L and L_c symmetric and negative definite. Let us define λ_i as the eigenvalues of $\mathcal{S} = \Omega_s - \Gamma_{s\to c}^T \Omega \Gamma_{s\to c}$. Then, if $\lambda_i > 0$, the iterative process converges.

Result 2

$\mathcal{S} = \Omega_s - \Gamma_{s\to c}^T \Omega \Gamma_{s\to c}$ being positive definite is equivalent to $\mathcal{C} = \Omega - \Gamma_{c\to s}^T \Omega_s \Gamma_{c\to s}$ being positive definite.

Theoretical results for Cartesian meshes

Result 3

Let $\Pi_{c\rightarrow s}$ be a general 1D interpolation matrix from cells to faces,

$$\Pi_{c\rightarrow s} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \tag{11}$$

Then, if $\boxed{V_r - \sum_{s=1}^{n} \sum_{j=x,y,z} \left| \sum_{l=1}^{m} a_{lr} V_l^{stagg} a_{ls} n_l^i n_l^j \right| \geq 0}$ $\forall r \in \{1, ..., n\}, i \in \{x, y, z\}$, the Poisson iterative process will converge.

Theoretical results for Cartesian meshes

It can be proven that the weighted volume scheme meet the previous criteria for all Cartesian meshes, with independence on the location of the cell-center (inside the control volume).

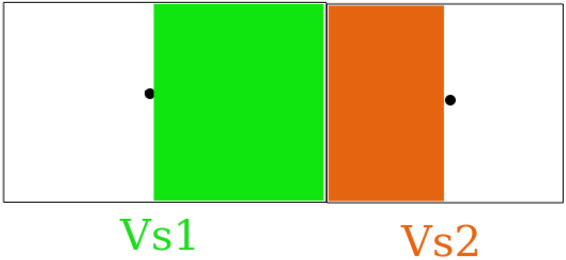

Figure 12: Staggered volumes

Invariance of definiteness

Motivation: It is easy to prove that a Cartesian mesh meet the previous criteria, but it is not easy to prove it for rotated Cartesian meshes.

Result 4

The definiteness of $\mathcal{C} = \Omega - \Gamma_{c\rightarrow s}^{T}\Omega_s\Gamma_{c\rightarrow s} = \Omega - (I_3 \otimes \Pi_{cs})^T N^T \Omega_s N (I_3 \otimes \Pi_{cs})$ is invariant under a change of basis.

- The face-normal vectors are contained in the matrix N.
- A change of basis should be a matrix which changes properly the matrix N.
- Interesting change of basis could be rotations or reflections.
- As N is inside $\Gamma_{c\rightarrow s}$, changing the basis will lead us to a new volumetric interpolator $\Gamma_{c\rightarrow s}^{new}$.

As no theoretical result could be found for unstructured meshes, a numerical experiment has been carried out:

- Checking the sign of the eigenvalues of $L - L_c$.
- Computing the global error and the rate of convergence of the new interpolator.

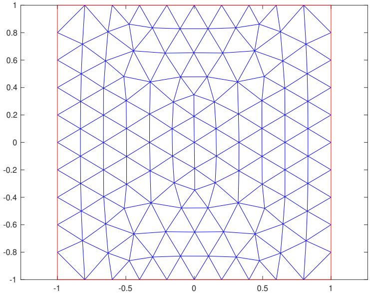

Figure 13: Mesh used to obtain the eigenvalues of $L - L_c$

Numerical results for unstructured meshes

Eigenvalues of $L - L_c$ obtained placing the volume center in the orthocenter.

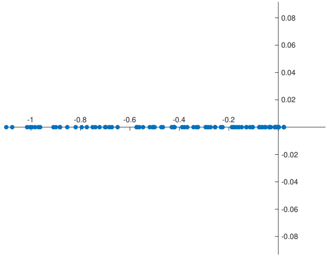

Figure 14: Eigenvalues of $L - L_c$ (mid-point interpolator)

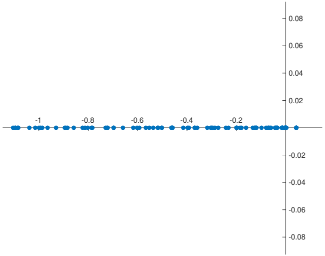

Figure 15: Eigenvalues of $L - L_c$ (volume weighted interpolator)

Numerical results for unstructured meshes

Eigenvalues of $L - L_c$ obtained placing the volume center in the circumcenter.

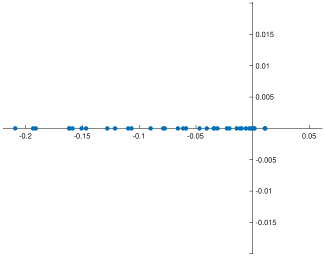

Figure 16: Eigenvalues of $L - L_c$ (mid-point interpolator)

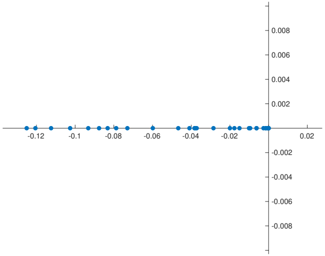

Figure 17: Eigenvalues of $L - L_c$ (volume weighted interpolator)

Numerical results for unstructured meshes

Problems and remarks:

- When the circumcenter is outside the triangle, positive eigenvalues are found.
- Possibility of introducing negative staggered volums has been tested, however, this makes L not to be negative definite and introduce more problems.
- Possible solution: working with acute-angled triangles because they have the circumcenter inside.

Numerical results for unstructured meshes

For computing the global error and the rate of convergence,

- 60 random meshes have been used.
- Average step-size is defined as $trace(\Omega_c)/n$, where n is the number of control volumes.
- Velocity was assumed to be $u = cos(2\pi x)sin(2\pi y)$ and $v = -sin(2\pi x)cos(2\pi y)$.
- The next function has been used to stretch the meshes:

$$x_i^{new} = \frac{1}{2}\left(1 + \frac{tanh(\gamma(2x_i - 1))}{tanh(\gamma)}\right)$$

The global error is understood as $||p^s - p^c||_\infty$, where:

- p^s is obtained solving $Lp^s = Mu_s$.
- p^c is obtained solving $Lp^c = M\Gamma_{c\to s}u_c$.

Numerical results for unstructured meshes

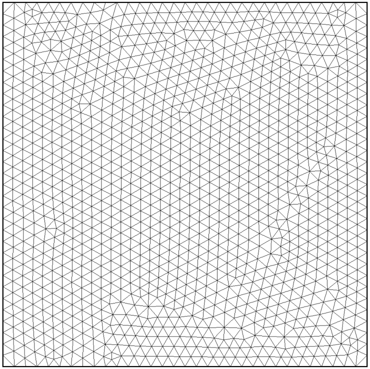

Figure 18: Example of mesh used with $\gamma = 0$.

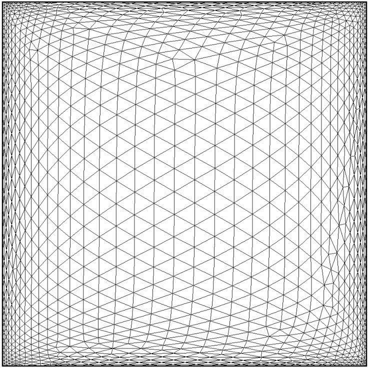

Figure 19: Example of mesh used with $\gamma = 2$.

Numerical results for unstructured meshes

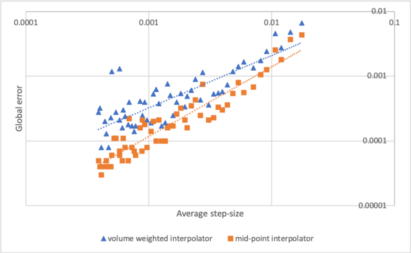

Figure 20: Mean global error results for the mesh with $\gamma = 0$.

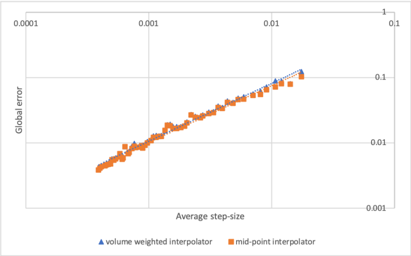

Figure 21: Mean global error results for the mesh with $\gamma = 2$.

Numerical results for unstructured meshes

The rate of convergence, p_{conv}, is found fitting the points assuming a relation $y = kx^{p_{conv}}$, where k is a constant.

γ	**0**	**0.5**	**1**	**1.5**	**2**
p **(volume weighted interpolator)**	0.81	0.96	1.00	1.05	0.88
p **(mid-point interpolator)**	1.07	0.97	0.98	1.02	0.87

- As we can see, the rate of convergence value is around 1 for both schemes.
- Furthermore, the order of the error found with both schemes is similar, and it becomes practically the same for stretched meshes.

Conclusions

- The appearance of unphysical velocities is a common problem found in highly distorted meshes.
- The volume weighted scheme solves this problem for Cartesian meshes.
- For unstructured triangular meshes, we can conclude that the global pressure errors are practically the same for both schemes but it seems that weighted volume scheme is more stable for distorted meshes.
- Apparently, the new scheme can not be extended for cases when the circumcenter is outside the triangle, at least by using negative staggered volumes, due to breaking the negative definiteness of L.