CTTC UPC
Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

Symmetry-preserving discretization of Navier-Stokes on unstructured grids: collocated vs staggered

F.Xavier Trias[1], Nicolás Valle[1], Andrey Gorobets[2], Assensi Oliva[1]

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8th. European Congress on Computational Methods in Applie Sciences and Engineering (ECCOMAS 2020)

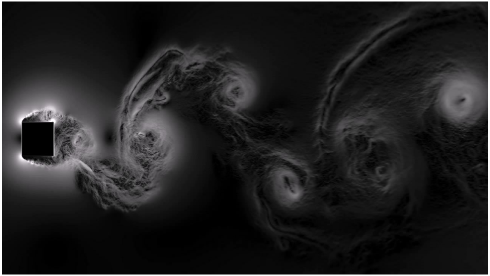
Virtual Congress 11-15 January, 2021

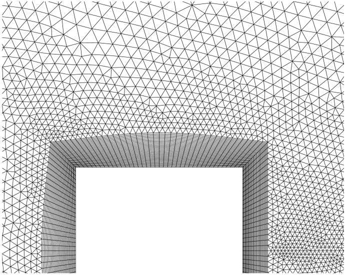
Contents

Motivation

Research question:

- Can we construct numerical discretizations of the Navier-Stokes equations suitable for **complex geometries**, such that the **symmetry properties** are exactly preserved?





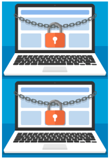
DNS[1] of the turbulent flow around a square cylinder at $Re = 22000$

[1] F.X.Trias, A.Gorobets, A.Oliva. *Turbulent flow around a square cylinder at Reynolds number 22000: a DNS study*, **Computers&Fluids**, 123:87-98, 2015.

Motivation

Frequently used general purpose CFD codes:

- STAR-CCM+
- ANSYS-FLUENT
- Code-Saturne
- OpenFOAM



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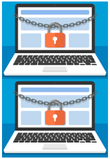
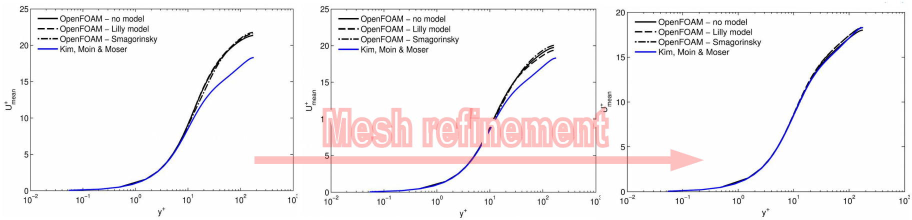


Main common characteristics of LES in such codes:

- **Unstructured finite volume** method, **collocated** grid
- Second-order spatial and temporal discretisation
- Eddy-viscosity type LES models

Motivation

OpenFOAM® LES[2] results of a turbulent channel for at $Re_\tau = 180$



$13\times76\times20$	$19\times78\times28$	$38\times78\times57$
$\Delta x^+=90, \Delta y^+_{wall}=0.5, \Delta z^+=30$	$\Delta x^+=60, \Delta y^+_{wall}=0.5, \Delta z^+=20$	$\Delta x^+=30, \Delta y^+_{wall}=0.5, \Delta z^+=10$

[2] E.M.J.Komen, L.H.Camilo, A.Shams, B.J.Geurts, B.Koren. *A quantification method for numerical dissipation in quasi-DNS and under-resolved DNS, and effects of numerical dissipation in quasi-DNS and under-resolved DNS of turbulent channel flows*, **Journal of Computational Physics**, 345, 565-595, 2017.

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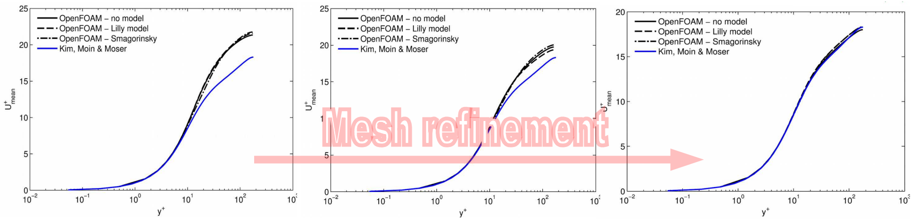

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- Are LES results are merit of the SGS model?

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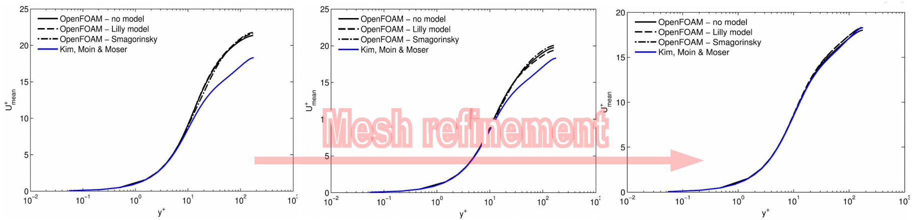

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- Are LES results are merit of the SGS model? Apparently **NOT**!!! ✗

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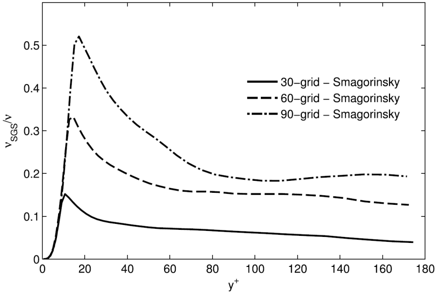

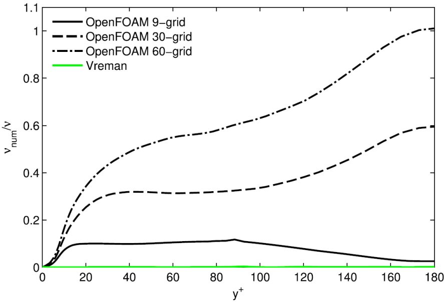

$$\nu_{num} \neq 0$$

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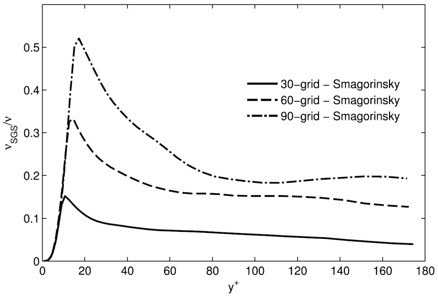

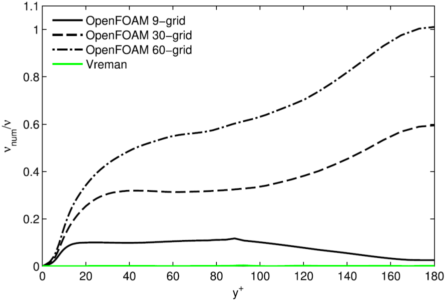

$$\nu_{SGS} < \nu_{num} \neq 0$$

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Symmetry-preserving discretization

Continuous

$$\frac{\partial \boldsymbol{u}}{\partial t} + C(\boldsymbol{u}, \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u} - \nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Symmetry-preserving discretization

Continuous	Discrete
$\frac{\partial \boldsymbol{u}}{\partial t} + C(\boldsymbol{u}, \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u} - \nabla p$	$\Omega \frac{d\boldsymbol{u}_h}{dt} + \mathsf{C}(\boldsymbol{u}_h)\,\boldsymbol{u}_h = \mathsf{D}\boldsymbol{u}_h - \mathsf{G}\boldsymbol{p}_h$
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$\langle C(\boldsymbol{u}, \varphi_1), \varphi_2 \rangle = - \langle C(\boldsymbol{u}, \varphi_2), \varphi_1 \rangle$	$\mathsf{C}(\boldsymbol{u}_h) = -\mathsf{C}^T(\boldsymbol{u}_h)$

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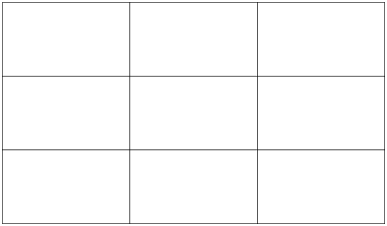
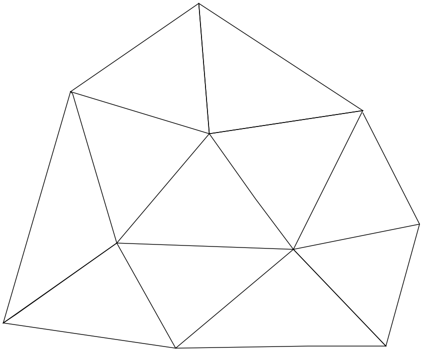
$\left\langle \nabla^2 \boldsymbol{a}, \boldsymbol{b} \right\rangle = \left\langle \boldsymbol{a}, \nabla^2 \boldsymbol{b} \right\rangle$	$\mathsf{D} = \mathsf{D}^T \quad def -$

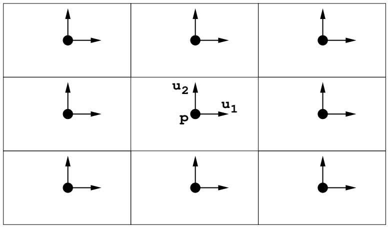
Collocated vs staggered

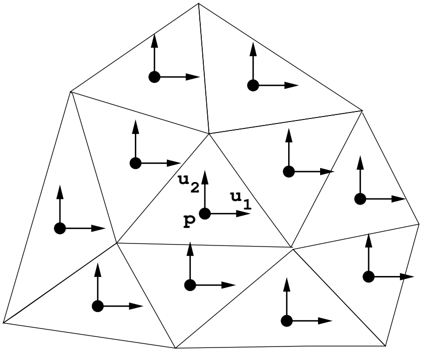




Collocated vs staggered

Collocated





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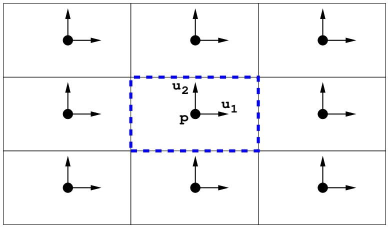

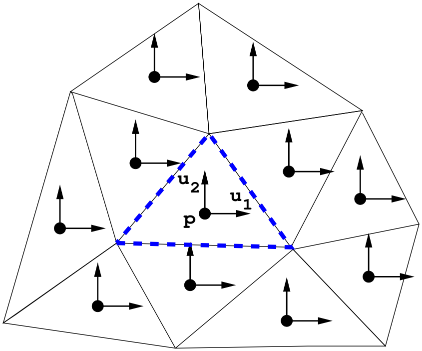

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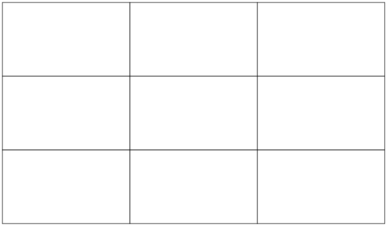
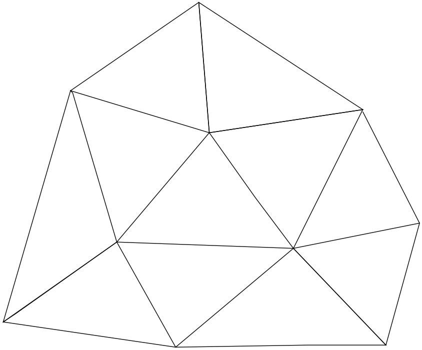
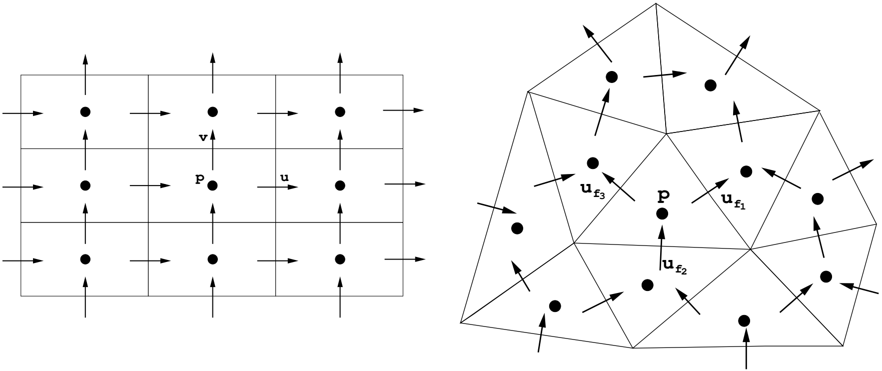




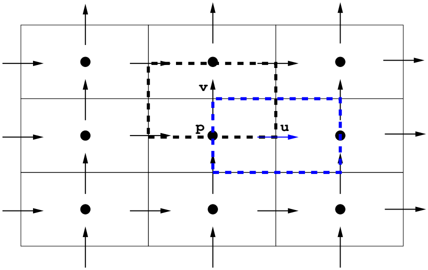
Collocated vs staggered

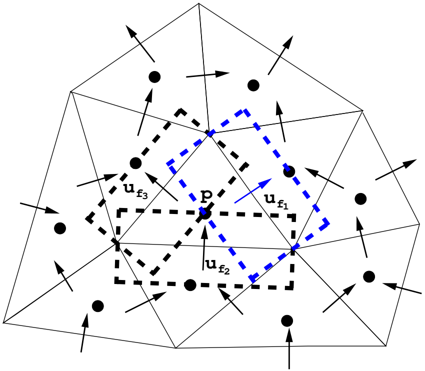
Staggered



Collocated vs staggered

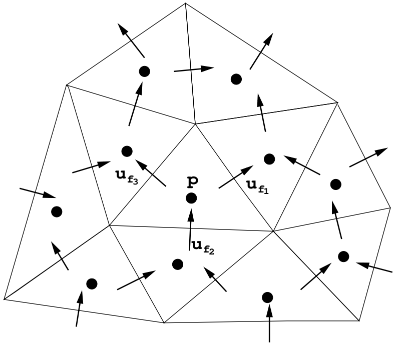
Staggered





Why staggered?

$$\Omega_s \frac{d\boldsymbol{u}_s}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_s = \mathsf{D}\boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c; \quad \mathsf{M}\boldsymbol{u}_s = \mathbf{0}_c$$



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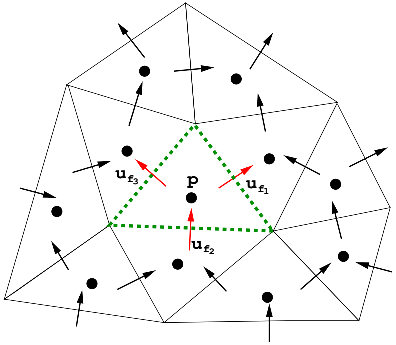
Let's consider we have $\boldsymbol{u}_s$ such as

$$\mathsf{M}\boldsymbol{u}_s \neq \mathbf{0}_c$$



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Let's consider we have $\boldsymbol{u}_s$ such as

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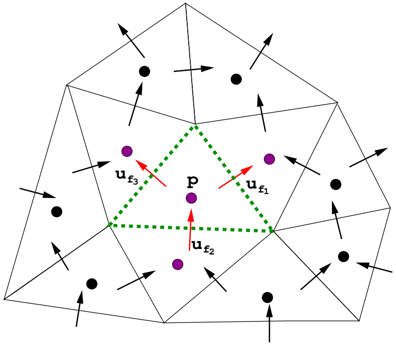
then, we can easily project $\boldsymbol{u}_s$

$$\boldsymbol{u}_s = \boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c$$



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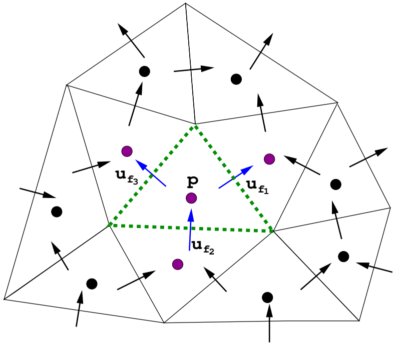

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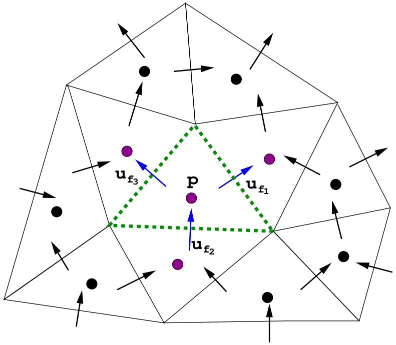
Finally, this leads to a Poisson eq.

$$\mathsf{M}\mathsf{G}\boldsymbol{p}_c = \mathsf{M}\boldsymbol{u}_s$$



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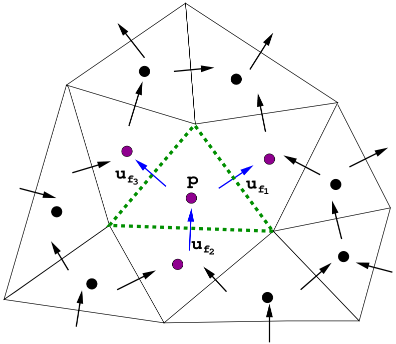
If $\Omega_s \mathsf{G} = -\mathsf{M}^T$

$$\langle \nabla\cdot\boldsymbol{a}, \varphi \rangle = -\langle \boldsymbol{a}, \nabla\varphi \rangle$$



Why staggered? Everything seems to be in the right place!

$$\Omega_s \frac{d\boldsymbol{u}_s}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_s = \mathsf{D}\boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c; \quad \mathsf{M}\boldsymbol{u}_s = \mathbf{0}_c$$

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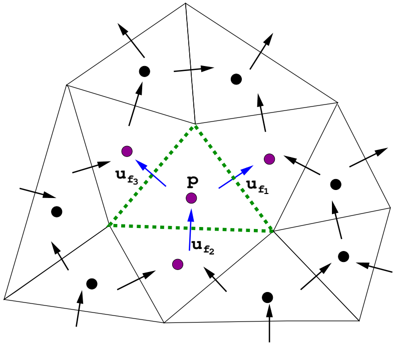
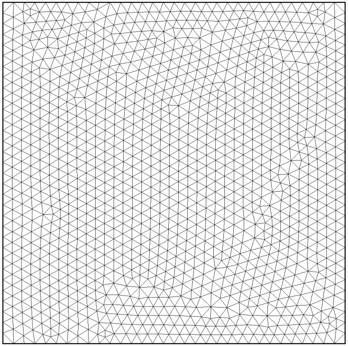


$$\text{If } \Omega_s\mathsf{G} = -\mathsf{M}^T \implies \langle \boldsymbol{u}_s, \mathsf{G}\boldsymbol{p}_c\rangle_h = \boldsymbol{u}_s^T\Omega_s\mathsf{G}\boldsymbol{p}_c = -(\mathsf{M}\boldsymbol{u}_s)^T\boldsymbol{p}_c = 0$$

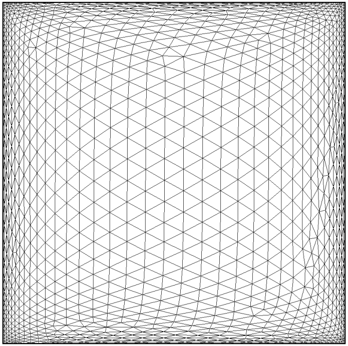
$$\langle \nabla\cdot\boldsymbol{a}, \varphi\rangle = -\langle \boldsymbol{a}, \nabla\varphi\rangle \implies \langle \boldsymbol{u}, \nabla p\rangle = -\langle \nabla\cdot\boldsymbol{u}, p\rangle = 0$$

But is this discrete Laplacian accurate?

Without stretching



With stretching



$$\nabla^2 \varphi = f(x,y) \quad \text{with } f(x,y) = \nabla^2 (k^{-2} \sin(kx) \sin(ky)) \text{ and } k = 25\pi$$

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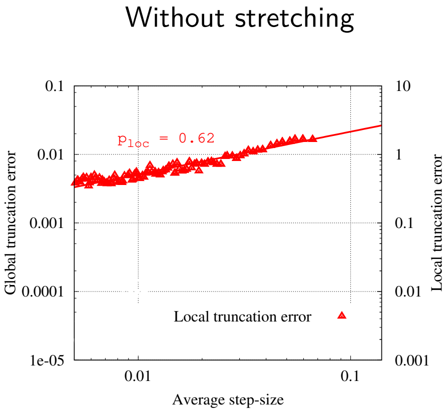

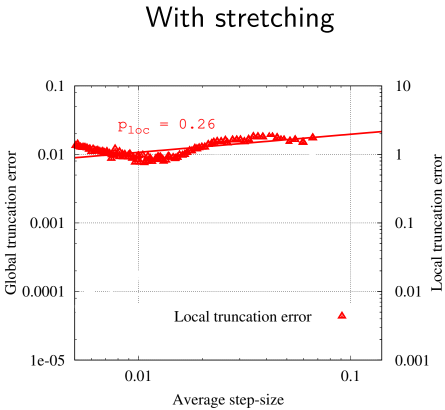

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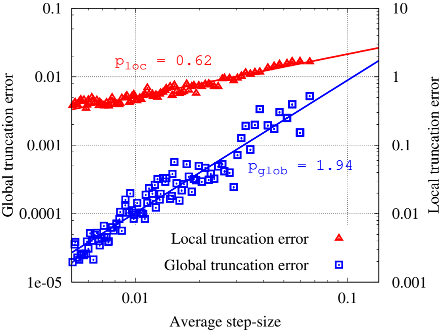
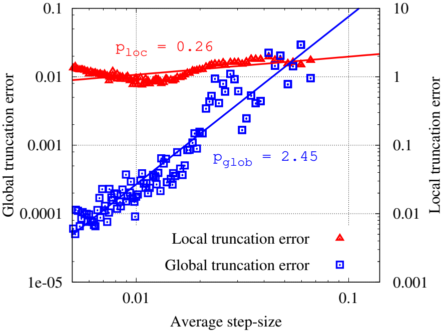
But is this discrete Laplacian accurate?

Yes, even for distorted unstructured meshes! And symmetries are preserved!

Without stretching



With stretching



$$\nabla^2\varphi = f(x,y) \quad \text{with } f(x,y) = \nabla^2(k^{-2}\sin(kx)\sin(ky)) \text{ and } k = 25\pi$$

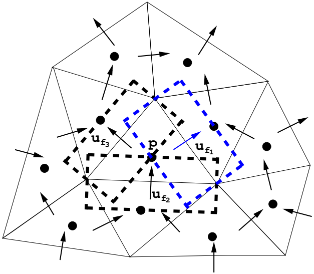
Then, why collocated arrangements are so popular?

- STAR-CCM+
- ANSYS-FLUENT
- Code-Saturne
- OpenFOAM

$$\Omega_s \frac{d\boldsymbol{u}_s}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_s = \mathsf{D}\boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c; \quad \mathsf{M}\boldsymbol{u}_s = \boldsymbol{0}_c$$

In staggered meshes

- p-$\boldsymbol{u}_s$ coupling is naturally solved ✓
- $\mathsf{C}\left(\boldsymbol{u}_s\right)$ and D difficult to discretize ✗



Then, why collocated arrangements are so popular?

- STAR-CCM+
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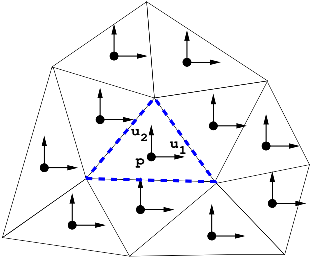
$$\Omega_c \frac{d\boldsymbol{u}_c}{dt} + \mathsf{C}(\boldsymbol{u}_s)\,\boldsymbol{u}_c = \mathsf{D}\boldsymbol{u}_c - \mathsf{G}_c \boldsymbol{p}_c; \quad \mathsf{M}_c \boldsymbol{u}_c = \boldsymbol{0}_c$$

In collocated meshes

- p-$\boldsymbol{u}_c$ coupling is cumbersome ✗
- $\mathsf{C}(\boldsymbol{u}_s)$ and D easy to discretize ✓
- Cheaper, less memory,... ✓



Then, why collocated arrangements are so popular?

Everything is easy except the pressure-velocity coupling...

- STAR-CCM+
- ANSYS-FLUENT
- Code-Saturne
- OpenFOAM

$$\Omega_c \frac{d\boldsymbol{u}_c}{dt} + \mathsf{C}(\boldsymbol{u}_s)\,\boldsymbol{u}_c = \mathsf{D}\boldsymbol{u}_c - \mathsf{G}_c\boldsymbol{p}_c; \quad \mathsf{M}_c\boldsymbol{u}_c = \boldsymbol{0}_c$$

In <u>collocated meshes</u>

- p-$\boldsymbol{u}_c$ coupling is cumbersome ✗
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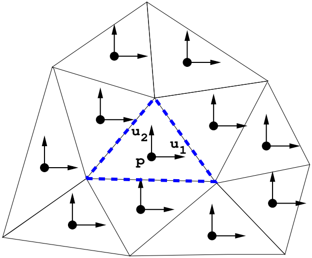
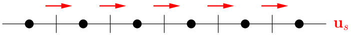


Pressure-velocity coupling on staggered grids

Works perfectly!



$\boldsymbol{u_s}$

Pressure-velocity coupling on staggered grids

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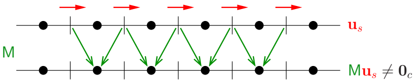

$\mathsf{M}\boldsymbol{u}_s$

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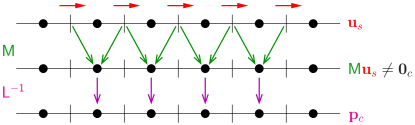

$$\underbrace{\mathsf{L}^{-1}\mathsf{M}\boldsymbol{u}_s}_{\boldsymbol{p}_c}$$

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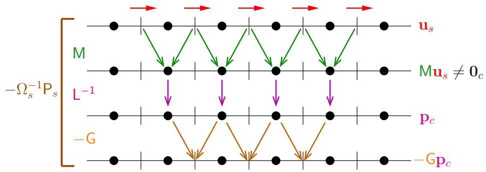

$$-\mathsf{G}\underbrace{\mathsf{L}^{-1}\mathsf{M}\boldsymbol{u_s}}_{\boldsymbol{p_c}}$$

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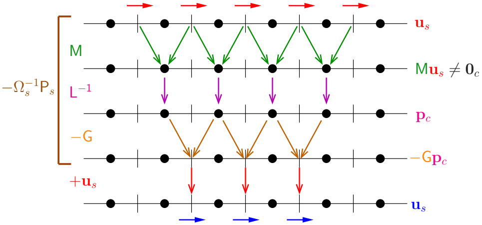

$$\boldsymbol{u}_s = \boldsymbol{u}_s - \mathsf{G}\underbrace{\mathsf{L}^{-1}\mathsf{M}\boldsymbol{u}_s}_{\boldsymbol{p}_c}$$

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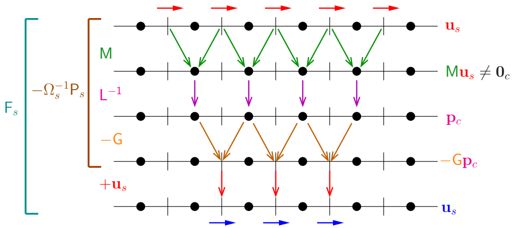

$$\boldsymbol{u}_s = \boldsymbol{u}_s - \mathsf{G}\underbrace{\mathsf{L}^{-1}\mathsf{M}\boldsymbol{u}_s}_{\boldsymbol{p}_c} = (I - \Omega_s^{-1}\mathsf{P}_s)\boldsymbol{u}_s = \mathsf{F}_s\boldsymbol{u}_s$$

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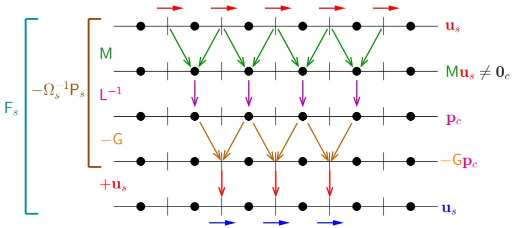

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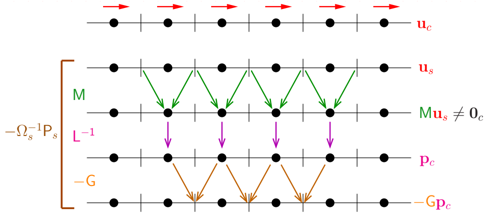
Pressure-velocity coupling on collocated grids

A vicious circle that cannot be broken...



$\boldsymbol{u}_c$

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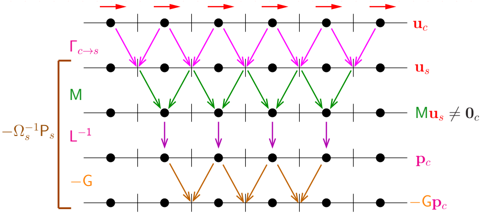

$\Gamma_{c\to s}\boldsymbol{u}_c$

Pressure-velocity coupling on collocated grids

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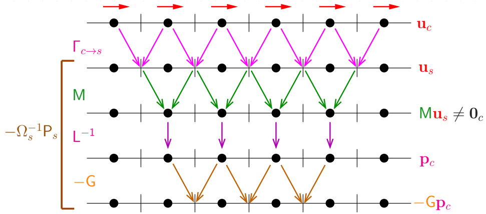

$$\mathsf{G}\mathsf{L}^{-1}\mathsf{M}\Gamma_{c\to s}\boldsymbol{u}_c$$

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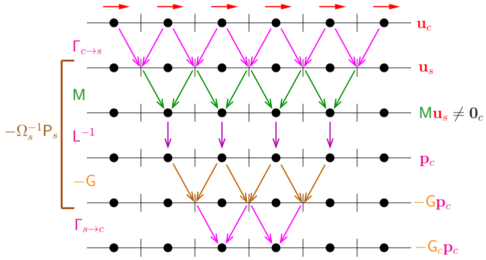

$$-\Gamma_{s\to c}\mathsf{G}\mathsf{L}^{-1}\mathsf{M}\Gamma_{c\to s}\boldsymbol{u}_c$$

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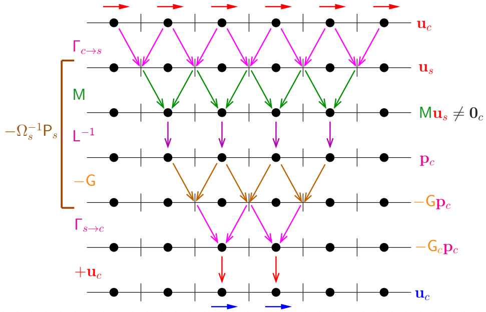

$$\boldsymbol{u}_c = \boldsymbol{u}_c - \Gamma_{s\to c}\mathsf{G}\mathsf{L}^{-1}\mathsf{M}\Gamma_{c\to s}\boldsymbol{u}_c$$

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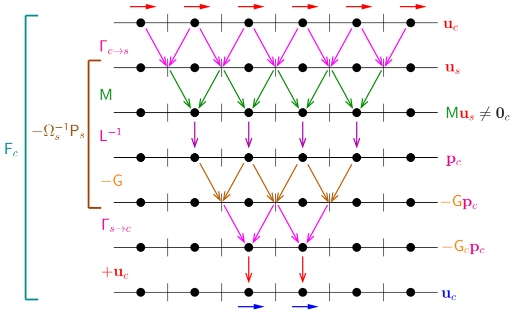

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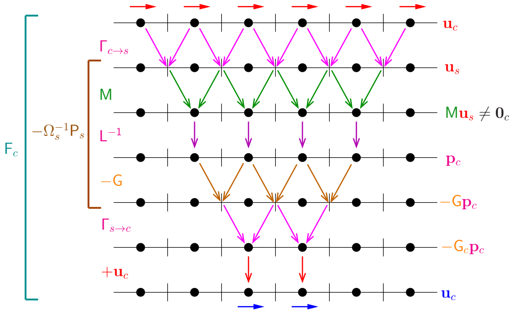

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To preserve symmetry we impose $\Gamma_{s\to c} = \Omega_c^{-1}\Gamma_{c\to s}^T\Omega_s$. This leads to

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where $\mathsf{L}_c = -\mathsf{M}\Gamma_{c\to s}\Omega_c^{-1}\Gamma_{c\to s}^T\mathsf{M}$ (wide-stencil discrete Laplacian).

Pressure-velocity coupling on collocated grids

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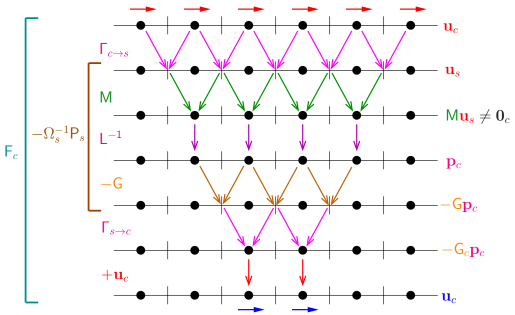

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Moreover, **contribution to kinetic enegy**: $\boldsymbol{p}_c(\mathsf{L}-\mathsf{L}_c)\boldsymbol{p}_c \neq 0$ ✗

Pressure-velocity coupling on collocated grids

A vicious circle that cannot be broken...

In summary[4]:

- Mass: $M\Gamma_{c\rightarrow s}\boldsymbol{u}_c = M\Gamma_{c\rightarrow s}\boldsymbol{u}_c - L_c L^{-1} M\Gamma_{c\rightarrow s}\boldsymbol{u}_c \approx \mathbf{0}_c$ ✗
- Energy: $\boldsymbol{p}_c \left(L - L_c\right) \boldsymbol{p}_c \neq 0$ ✗

[4] F.X.Trias, O.Lehmkuhl, A.Oliva, C.D.Pérez-Segarra, R.W.C.P.Verstappen. *Symmetry-preserving discretization of Navier-Stokes equations on collocated unstructured grids*, **Journal of Computational Physics**, 258 (1): 246-267, 2014.

Pressure-velocity coupling on collocated grids

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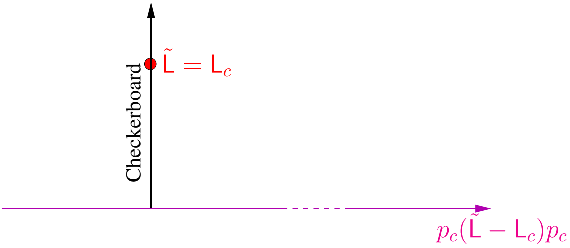

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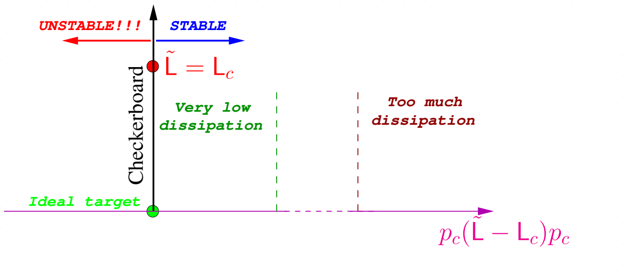

[4] D.Santos, J.Muela, N.Valle, F.X.Trias. *On the interpolation problem for the Poisson equation on collocated meshes*, **ECCOMAS2020**. Don't miss it!

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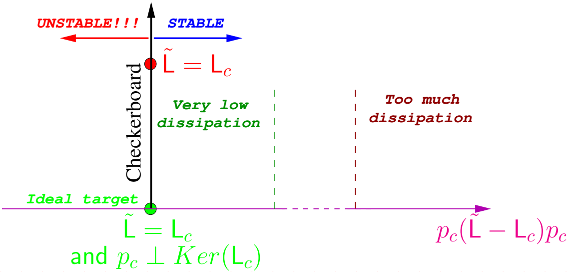

[4] Shashank, J.Larsson, G.Iaccarino. *A co-located incompressible Navier-Stokes solver with exact mass, momentum and kinetic energy conservation in the inviscid limit*, **Journal of Computational Physics**, 229: 4425-4430,2010.

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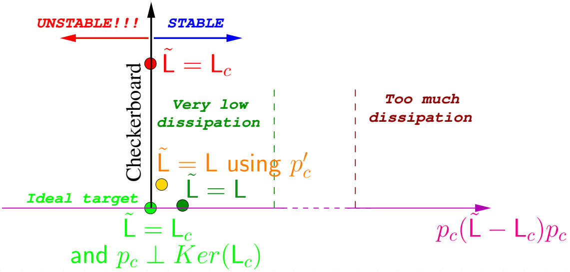

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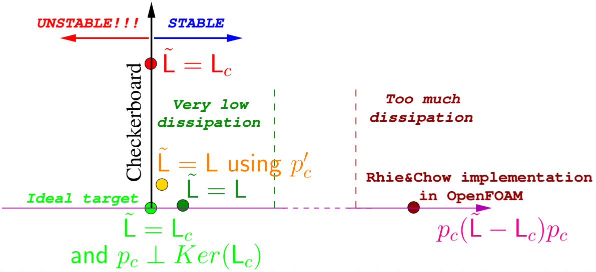
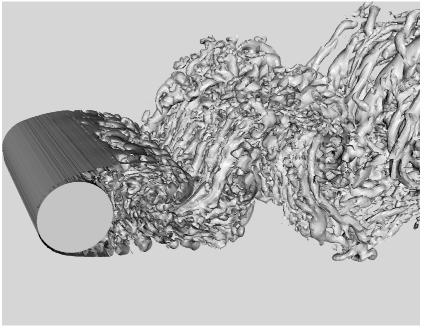


[4] E.Komen, J.A.Hopman, E.M.A.Frederix, F.X.Trias, R.W.C.P.Verstappen. *Energy-preserving discretisation for LES/DNS with unstructured collocated grids in OpenFOAM*, **ECCOMAS2020**. Don't miss it!

Pressure-velocity coupling on collocated grids

Examples of simulations

Despite these inherent limitations, symmetry-preserving collocated formulation has been successfully used for DNS/LES simulations[5]:



[5] R.Borrell, O.Lehmkuhl, F.X.Trias, A.Oliva. *Parallel Direct Poisson solver for discretizations with one Fourier diagonalizable direction*. **Journal of Computational Physics**, 230:4723-4741, 2011.

Pressure-velocity coupling on collocated grids

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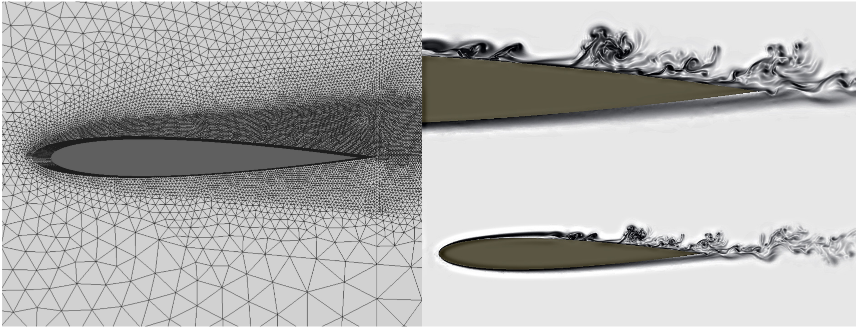
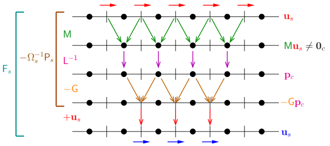
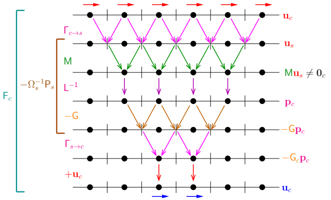


[5]F.X.Trias and O.Lehmkuhl. *A self-adaptive strategy for the time-integration of Navier-Stokes equations*. **Numerical Heat Transfer, part B**, 60(2):116-134, 2011.

Are staggered and collocated so different at the end?





Collocated: $$\boldsymbol{u}_c^{n+1} = \underbrace{\left(I_c - \Gamma_{s\to c}\Omega_s^{-1}\mathsf{P}_s\Gamma_{c\to s}\right)}_{\mathsf{F}_c} \underbrace{\overbrace{\left[I_c + \partial_t^c\right]\boldsymbol{u}_c^n}^{\boldsymbol{u}_c^p}}_{\mathsf{T}_c} = \underbrace{\mathsf{F}_c\mathsf{T}_c}_{NS_c}\,\boldsymbol{u}_c^n$$

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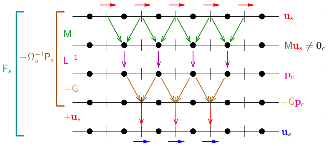

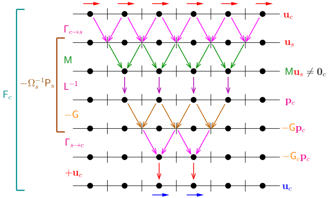

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Naïve collocated: $\boldsymbol{u}_c^{n+1} = \underbrace{\Gamma_{s\to c}\mathsf{F}_s\Gamma_{c\to s}\mathsf{T}_c}_{\widetilde{NS}_c} \boldsymbol{u}_c^n$

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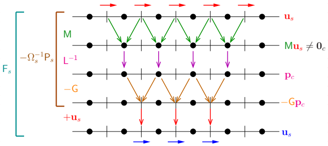

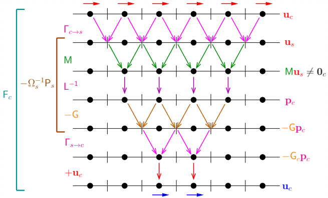

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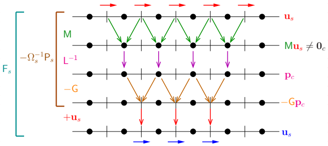
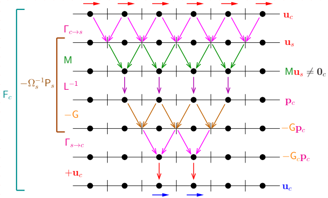
Naïve collocated: $$\boldsymbol{u}_c^{n+1} = \underbrace{\Gamma_{s\to c}\mathsf{F}_s\Gamma_{c\to s}\mathsf{T}_c}_{\widetilde{NS}_c} \boldsymbol{u}_c^n$$

Naïve staggered: $$\boldsymbol{u}_s^{n+1} = \underbrace{\mathsf{F}_s\Gamma_{c\to s}\mathsf{T}_c\Gamma_{s\to c}}_{\widetilde{NS}_s} \boldsymbol{u}_s^n$$

Are staggered and collocated so different at the end?





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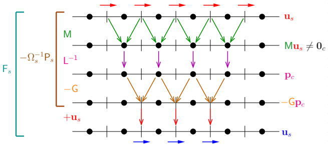
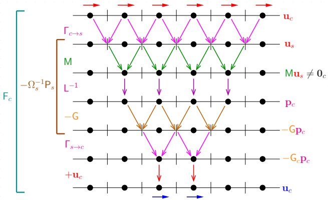
Naïve collocated: $$\boldsymbol{u}_c^{n+1} = \underbrace{\Gamma_{s\to c}\mathsf{F}_s\Gamma_{c\to s}\mathsf{T}_c}_{\widetilde{NS}_c} \boldsymbol{u}_c^n$$

Naïve staggered: $$\boldsymbol{u}_s^{n+1} = \underbrace{\mathsf{F}_s\Gamma_{c\to s}\mathsf{T}_c\Gamma_{s\to c}}_{\widetilde{NS}_s} \boldsymbol{u}_s^n$$

$$\boxed{\Gamma_{s\to c}(\widetilde{NS}_s)^n = (\widetilde{NS}_c)^n\Gamma_{c\to s}}$$

Are staggered and collocated so different at the end?





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Can we have a staggered formulation based only on collocated operators?

Then, it could be easily implemented in existing collocated codes such as OpenFOAM

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Similar approaches have been proposed in the literature before[6,7,8,9,10].

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[6] B.Perot. *Conservative properties of unstructured staggered meshs chemes*. **Journal of Comp. Physics**, 159: 58-89, 2000

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[8] K.Mahesh, G.Constantinescu, P.Moin. *A numerical method for large-eddy simulation in complex geometries*. **Journal of Computational Physics**,197: 215-240, 2004.

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Similar approaches have been proposed in the literature before[6,7,8,9,10].

Research question: then, why at the end collocated approach seems to be the winner?

- STAR-CCM+ (CD-adapco, SIEMENS)
- ANSYS-FLUENT (ANSYS FLUENT)
- Code-Saturne (Code Saturne, EDF, GPL)
- OpenFOAM (OpenFOAM®, GPL v3)

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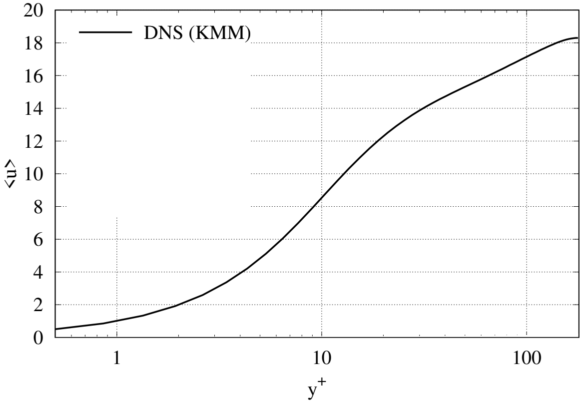
Results for a turbulent channel flow at $Re_\tau = 180$

Staggered C_s^{0}': $$\boldsymbol{u}_s^{n+1} = \underbrace{\left(I_s - \Omega_s^{-1} \mathsf{P}_s\right)}_{\mathsf{F}_s} \underbrace{\overbrace{\left[I_s + \Gamma_{c\to s} \partial_t^c \Gamma_{s\to c}\right] \boldsymbol{u}_s}^{\boldsymbol{u}_s^p}}_{\mathsf{T}_s}$$



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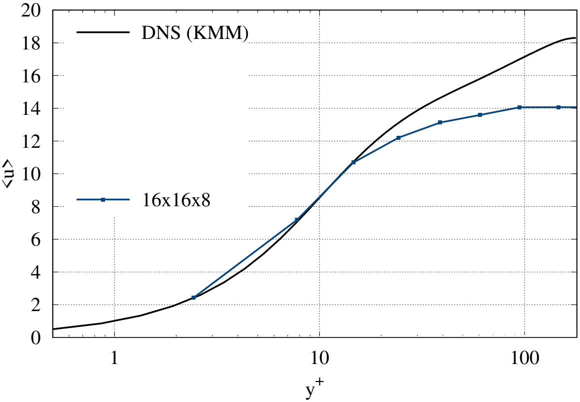

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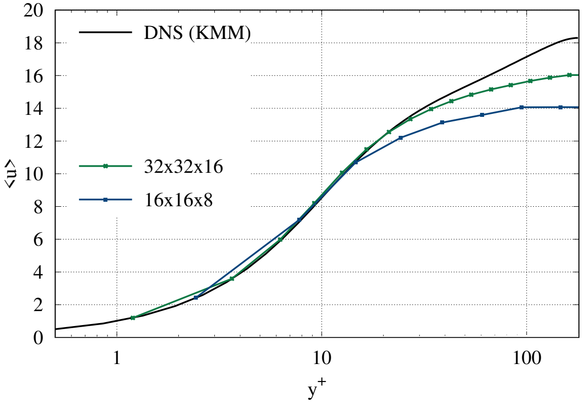

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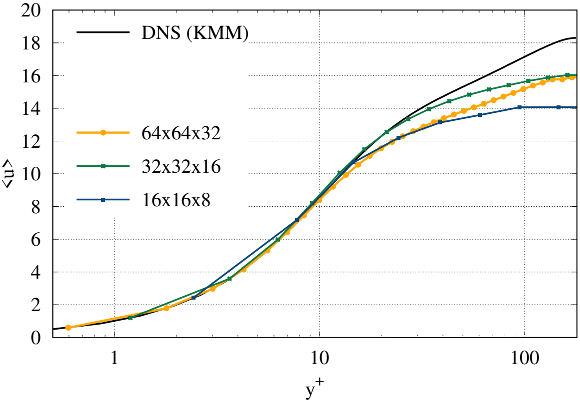

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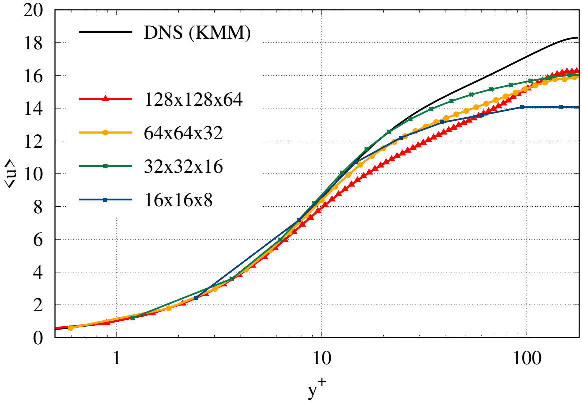

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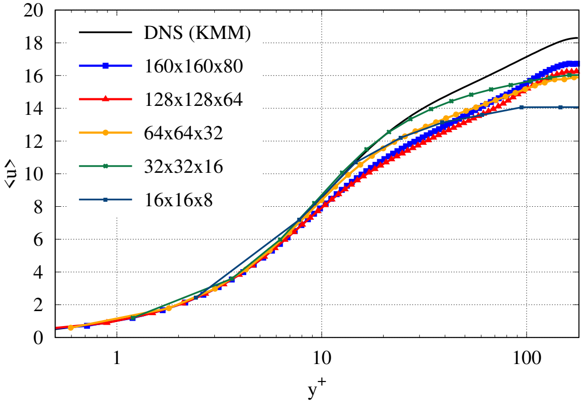

Dispersion errors analysis

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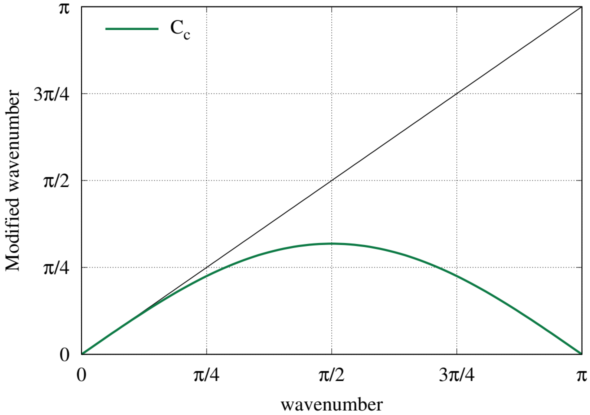

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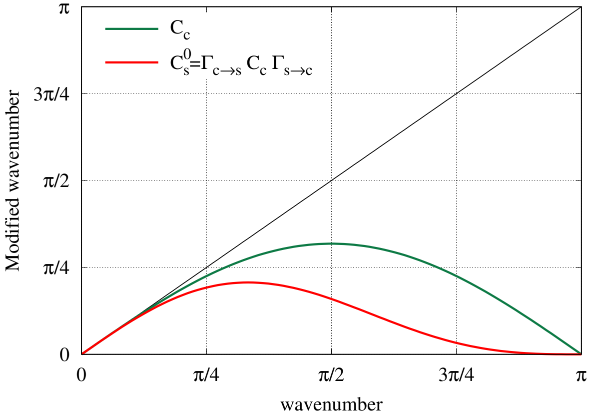

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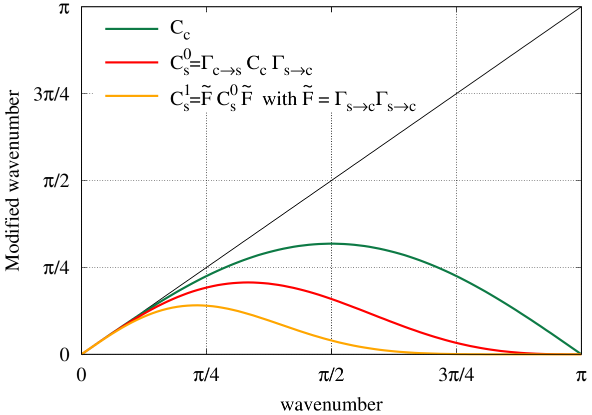
$$\text{Staggered } C_s^{1}\text{'}: \quad \boldsymbol{u}_s^{n+1} = \underbrace{(I_s - \Omega_s^{-1} \mathsf{P}_s)}_{\mathsf{F}_s} \underbrace{\overbrace{\left[I_s + \widetilde{\mathsf{F}} \Gamma_{c\to s} \partial_t^c \Gamma_{s\to c} \widetilde{\mathsf{F}} \right] \boldsymbol{u}_s}^{\boldsymbol{u}_s^p}}_{\mathsf{T}_s}$$

Filter: $\widetilde{\mathsf{F}} = \Gamma_{s\to c}\Gamma_{c\to s}$ ($\widetilde{\mathsf{F}} = \widetilde{\mathsf{F}}^T$ and positive semi-definite)



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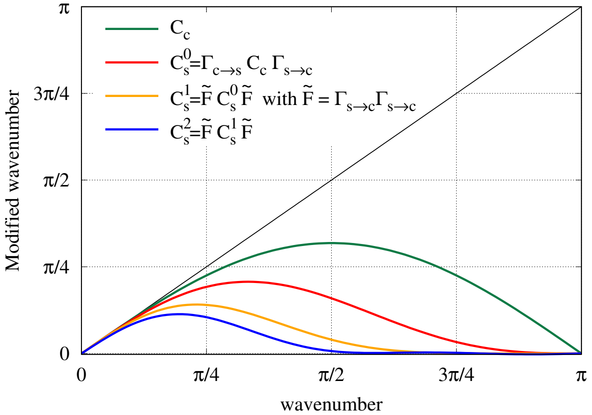

Results for a turbulent channel flow at $Re_\tau = 180$

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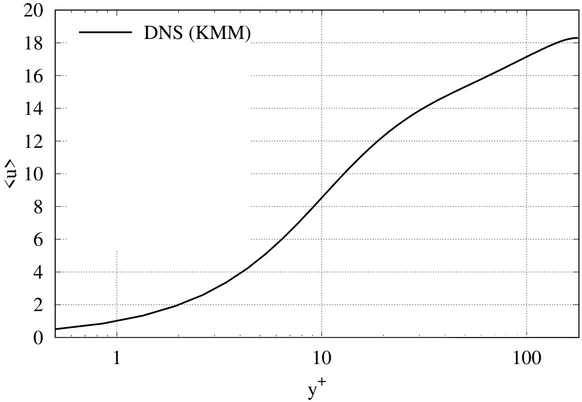

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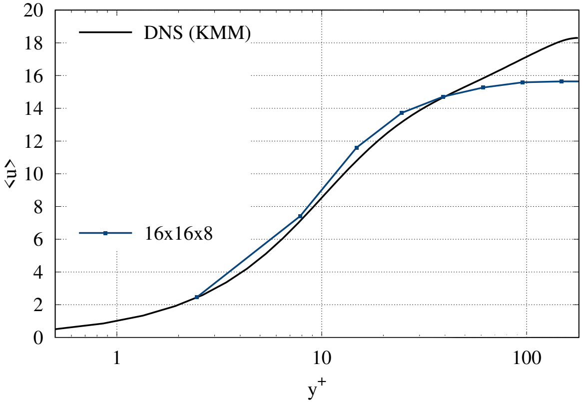

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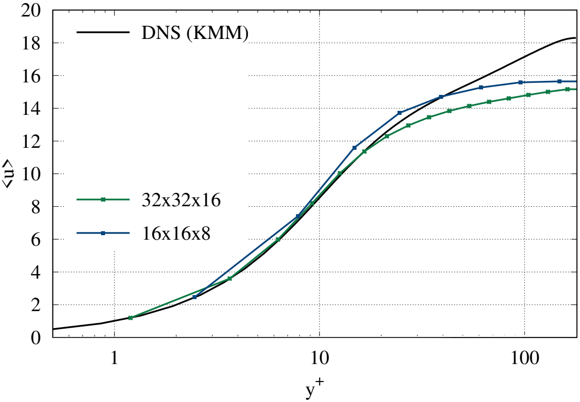

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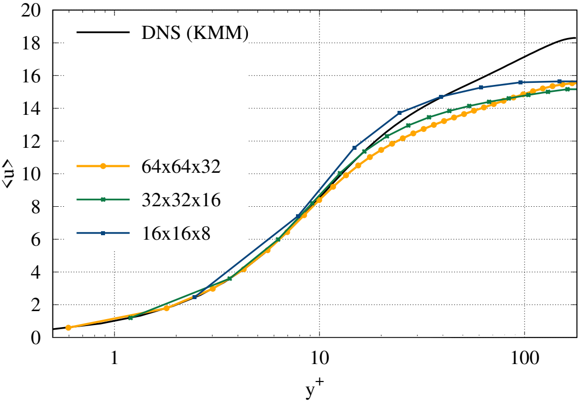

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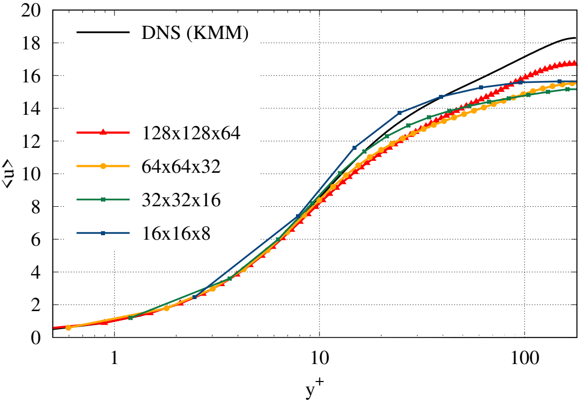

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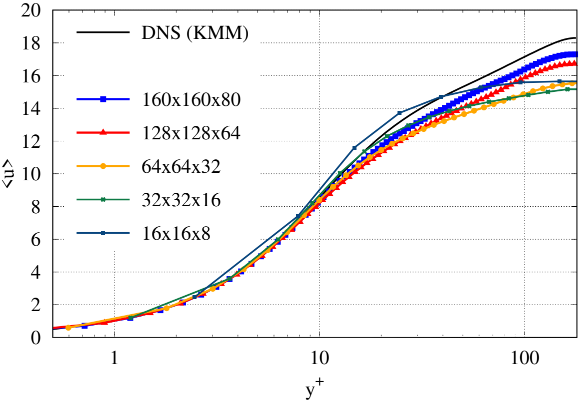

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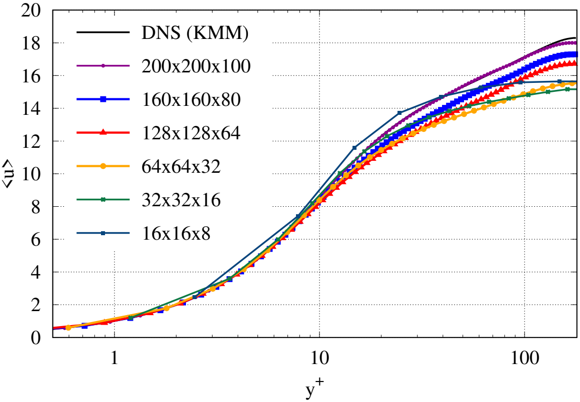
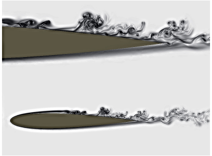


Concluding remarks

- **Preserving symmetries** either using staggered or collocated formulations is the key point for **reliable LES/DNS** simulations.



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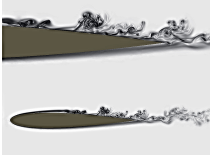

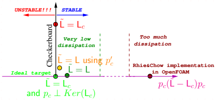

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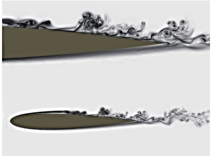
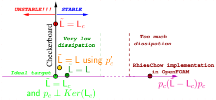
- Despite this, the CFD community have generally adopted collocated formulations due to the inherent **difficulties** to formulate a simple and robust **staggered discretization of momentum**.
  $\Longrightarrow$ A potential solution has been presented here...





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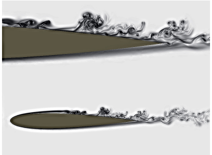

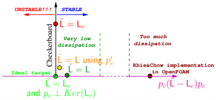

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Future research:

- Complete the analysis for higher Re_τ (running now...)
- Test for complex geometries using unstructured meshes (on-going)

Thank you for your virtual attendance