Symmetry-preserving discretisation methods for magnetohydrodynamics

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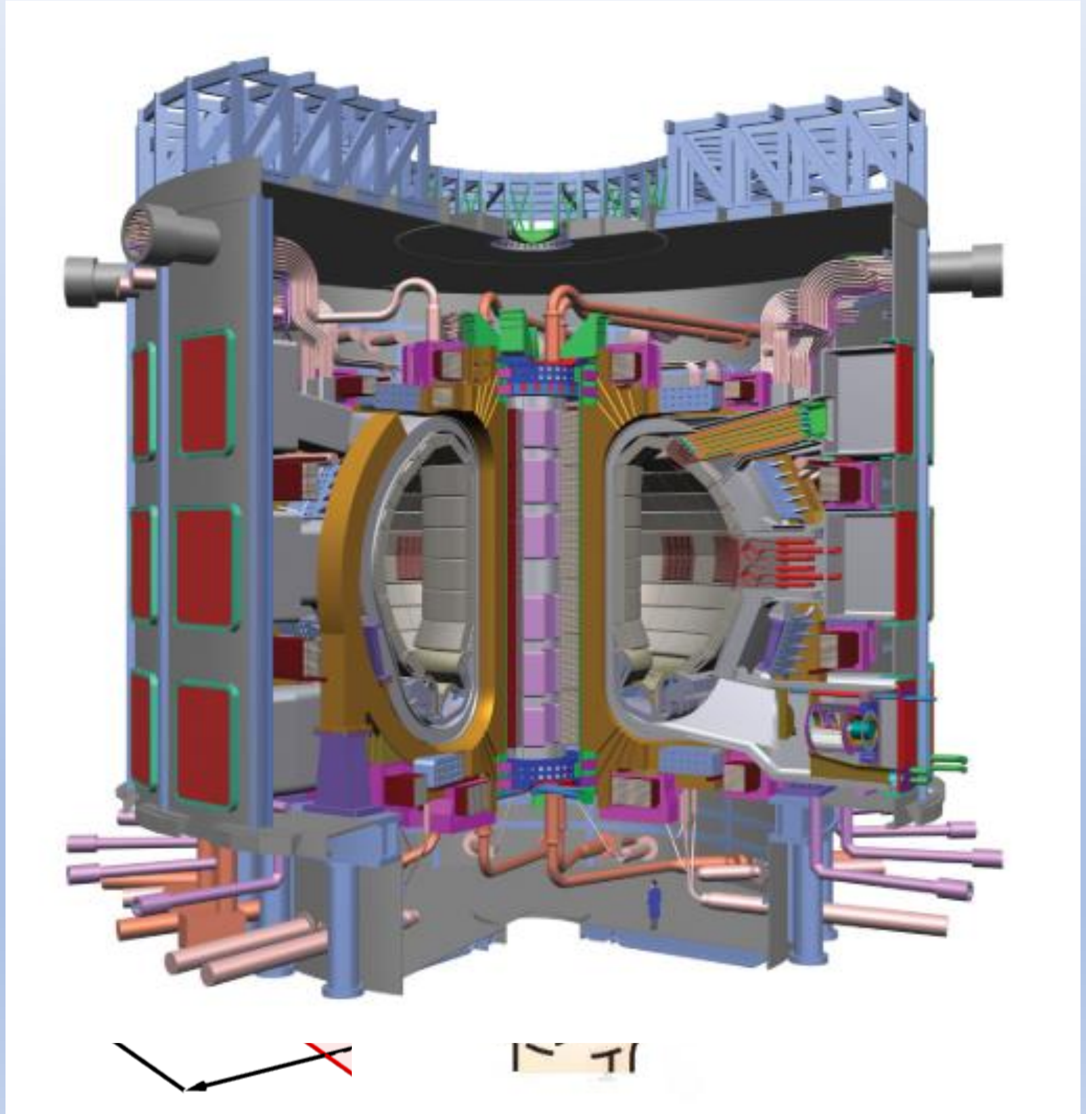
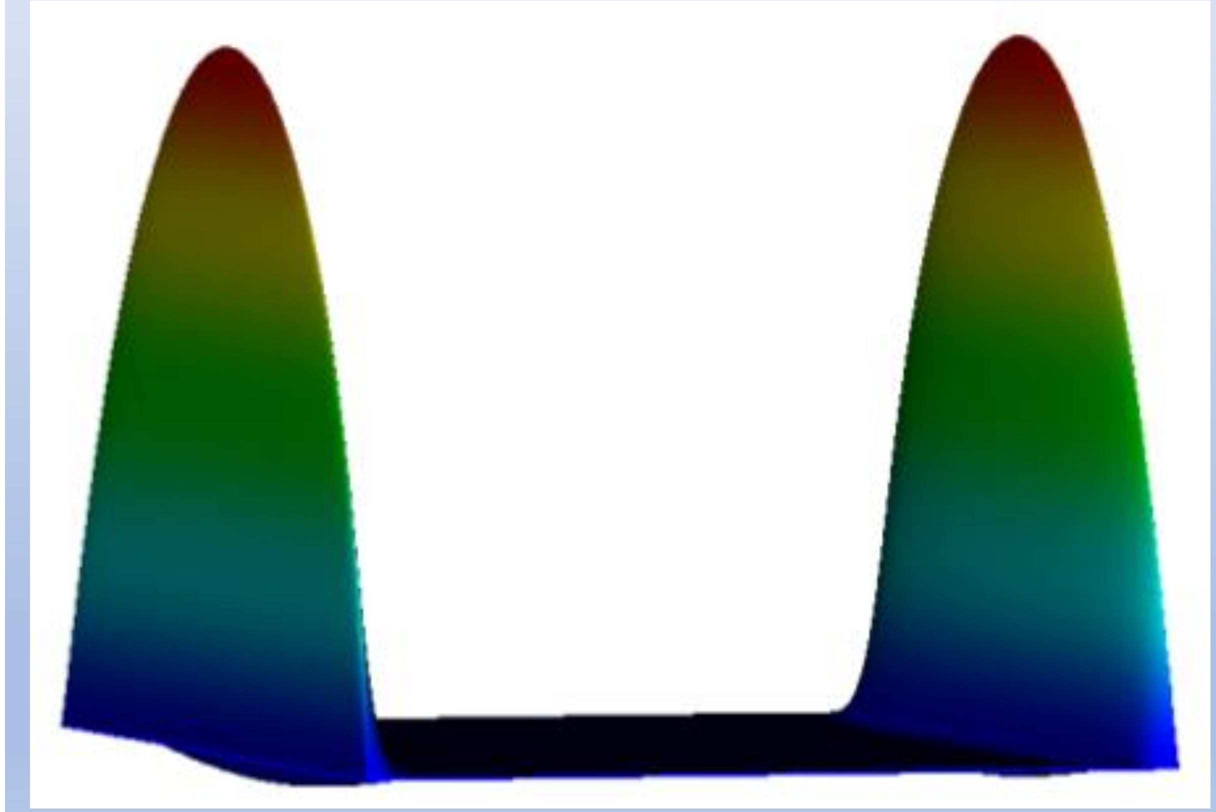
Conclusions

- Symmetry preserving (SyPr) method extended to include magnetohydrodynamic (MHD) flows
  - Accuracy on Cartesian grids
  - Unconditional stability
- New benchmark case designed using a Taylor-Green vortex (TGV)

MHD introduction



Liquid metals in magnetic field



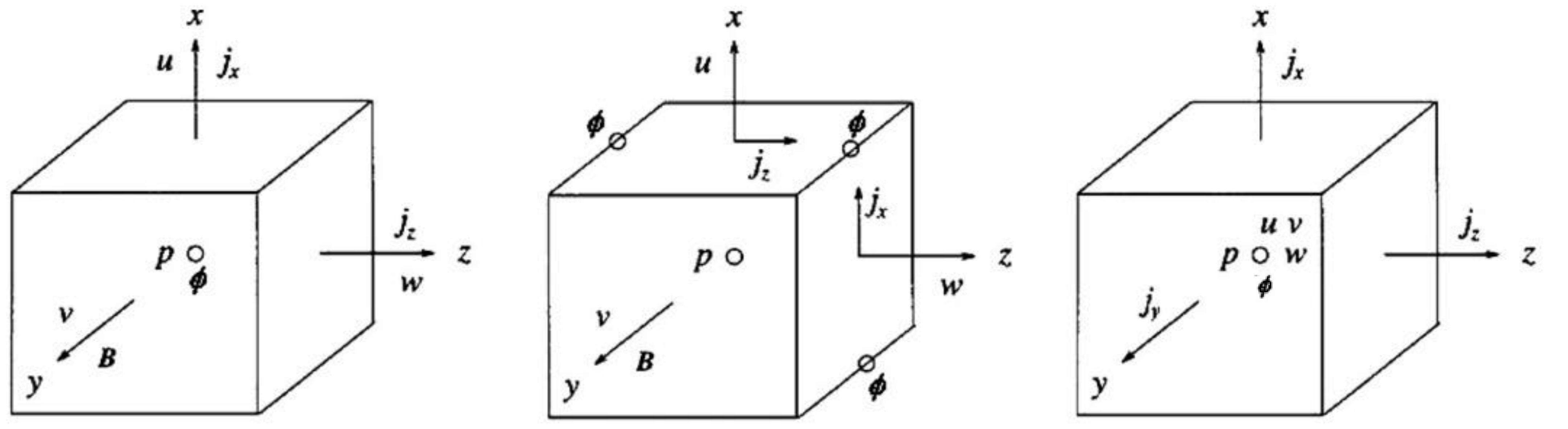
MHD challenge

- Balancing high forces
- Thin boundary layers
- Cross products & extra Poisson equation
  - Location of variables

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \nu\nabla^2\mathbf{u} - \nabla(p/\rho) + (\mathbf{J}\times\mathbf{B})/\rho$$

$$\mathbf{J} = \sigma(\mathbf{u}\times\mathbf{B} - \nabla\varphi)$$

$$\nabla^2\varphi = \nabla\cdot(\mathbf{u}\times\mathbf{B})$$



(a) staggered grid system (b) fully staggered grid system (c) collocated grid system

History of symmetry-preserving method

Conservative symmetric discretization - Verstappen & Veldman (2003)

Collocated unstructured grids - Trias *et al.* (2014)

Implementation into OpenFoam - Komen *et al.* (2021)

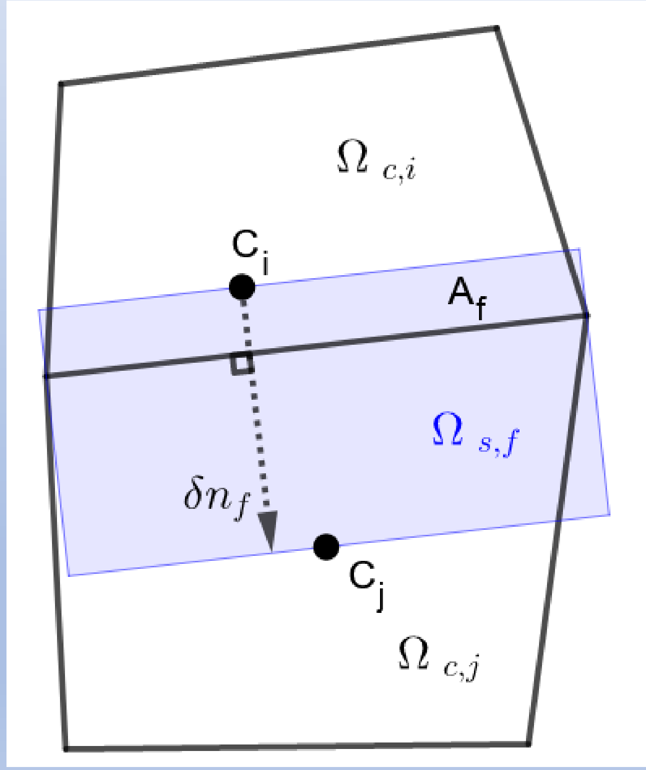
Extension to MHD

Symmetry-preserving method

Retaining symmetries in continuous operators

- Conserve energy + unconditionally stable

- Midpoint interpolation for convection
- Uncorrected gradient distances
- S→C interpolation: $\Gamma_{sc} = \Omega^{-1}\Gamma_{cs}^{T}\Omega_{s}$



Discretisation method

Method of Ni et al. 2007

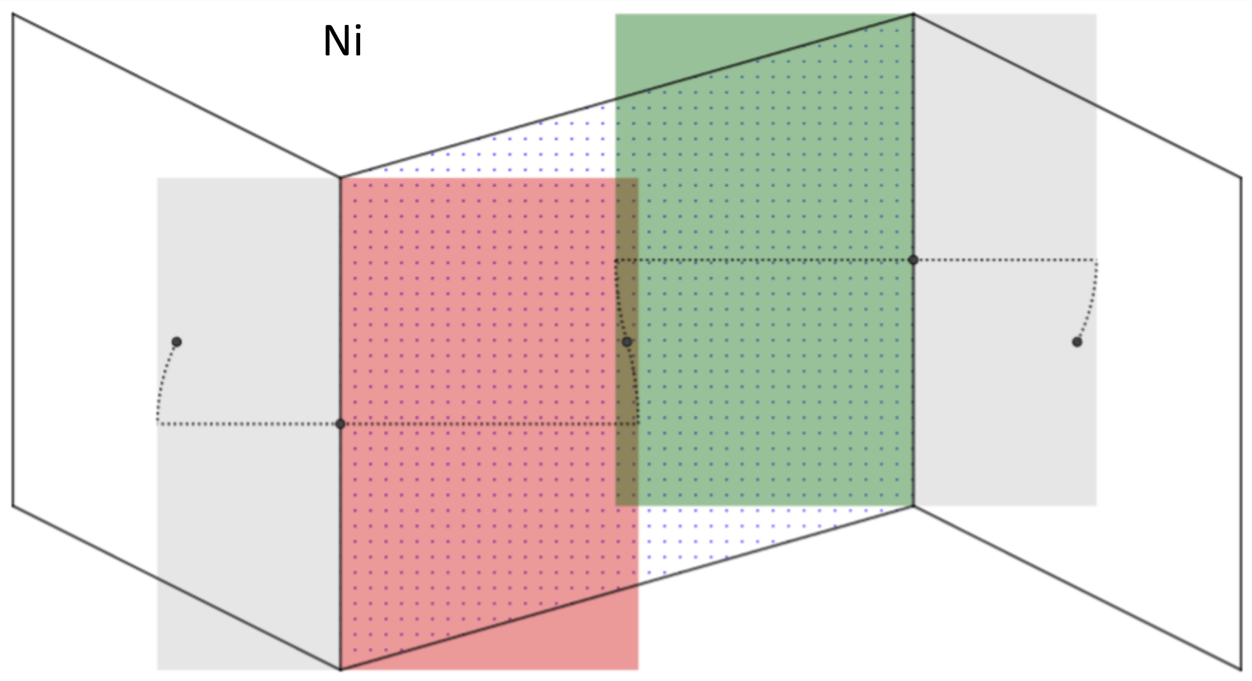
$\mathbf{u}_c^p = \mathbf{u}_c^n - \Delta t \Omega^{-1}(\mathrm{C}(\mathbf{u}_f^n) + \mathrm{D})\mathbf{u}_c^n + \frac{Ha^2}{Re}\mathbf{J}_c^n \times \mathbf{B}_c^n$	Predict velocity
$\mathbf{u}_f^p = \underline{\Gamma_{sc}}\mathbf{u}_c^p$	Interpolate
$\mathrm{L}\tilde{\mathbf{p}}_c^{n+1} = \mathrm{M}\mathbf{u}_f^p$	Pressure Poisson
$\mathbf{u}_f^{n+1} = \mathbf{u}_f^p - \mathrm{G}\tilde{\mathbf{p}}_c^{n+1}$	Correct flux
$\mathbf{u}_c^{n+1} = \mathbf{u}_c^p - \underline{\Gamma_{sc}}\mathrm{G}\tilde{\mathbf{p}}_c^{n+1}$	Update velocity
$\mathbf{J}_c^p = \mathbf{u}_c^{n+1} \times \mathbf{B}_c^{n+1}$	Predict current density
$\mathbf{J}_f^p = \underline{\Gamma_{sc}}\,\mathbf{J}_c^p$	Interpolate
$\mathrm{L}\boldsymbol{\varphi}_c^{n+1} = \mathrm{M}\mathbf{u}_f^p$	Electric potential Poisson
$\mathbf{J}_c^{n+1} = \underline{\Gamma_{sc}^{Ni}}\,(\mathbf{J}_f^p - \mathrm{G}\boldsymbol{\varphi}_c^{n+1})$	Update current density

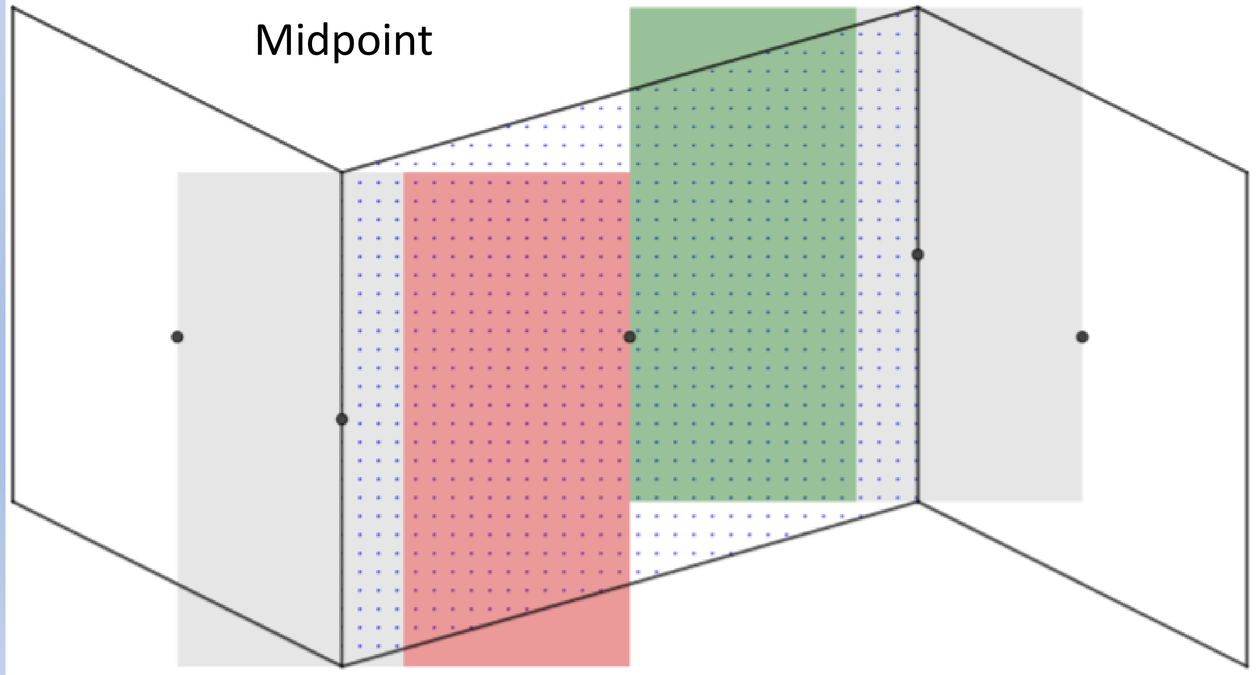
Ni interpolation (Γ_{sc}^{Ni}):

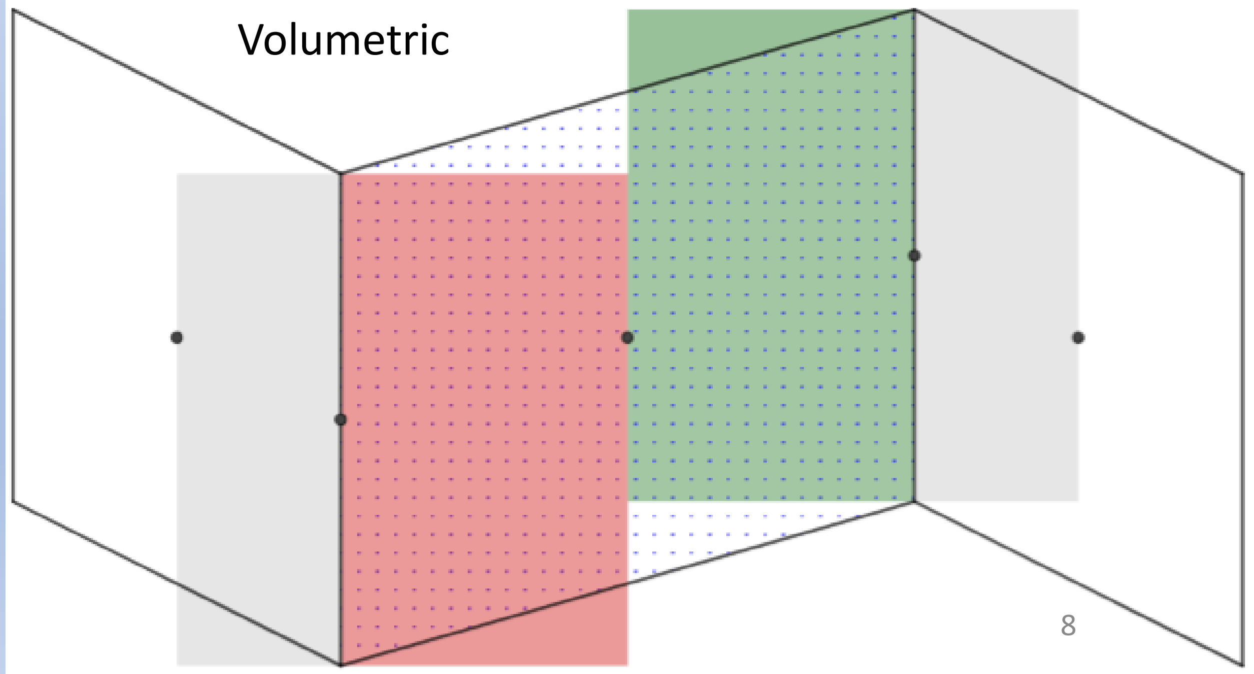
- Non-consistent: $\Gamma_{sc}^{Ni} \neq \Omega^{-1}\Gamma_{cs}^{T}\Omega_s$
- Non-conservative: $tr(\Omega_c) \neq tr(\Omega_s)$

Interpolation methods

1. Ni: $\frac{1}{[\Omega_c]_{i,i}} \sum_f J_f (\mathbf{r}_f - \mathbf{r}_i) s_f$
2. Midpoint
3. Volumetric







Discretisation method

Method of Ni et al. 2007

$$\mathbf{u}_c^p = \mathbf{u}_c^n - \Delta t \Omega^{-1}\left(\mathrm{C}\left(\mathbf{u}_f^n\right) + \mathrm{D}\right)\mathbf{u}_c^n$$

$$+\frac{Ha^2}{Re}\mathbf{J}_c^n \times \mathbf{B}_c^n$$

$$\mathbf{u}_f^p = \Gamma_{sc}\mathbf{u}_c^p$$

$$\mathrm{L}\tilde{\mathbf{p}}_c^{n+1} = \mathrm{M}\mathbf{u}_f^p$$

$$\mathbf{u}_f^{n+1} = \mathbf{u}_f^p - \mathrm{G}\tilde{\mathbf{p}}_c^{n+1}$$

$$\mathbf{u}_c^{n+1} = \mathbf{u}_c^p - \Gamma_{sc}\mathrm{G}\tilde{\mathbf{p}}_c^{n+1}$$

$$\mathbf{J}_c^p = \mathbf{u}_c^{n+1} \times \mathbf{B}_c^{n+1}$$

$$\mathbf{J}_f^p = \Gamma_{sc}\,\mathbf{J}_c^p$$

$$\mathrm{L}\boldsymbol{\varphi}_c^{n+1} = \mathrm{M}\mathbf{u}_f^p$$

$$\mathbf{J}_c^{n+1} = \Gamma_{sc}^{Ni}\left(\mathbf{J}_f^p - \mathrm{G}\boldsymbol{\varphi}_c^{n+1}\right)$$

Volumetric interpolation

Predictor fields

→

Symmetry preserving method

$$\mathbf{u}_c^p = \mathbf{u}_c^n - \Delta t \Omega^{-1}\left(\mathrm{C}\left(\mathbf{u}_f^n\right) + \mathrm{D}\right)\mathbf{u}_c^n$$

$$+\frac{Ha^2}{Re}\mathbf{J}_c^n \times \mathbf{B}_c^n - \underline{\Gamma_{sc}^{Vol}}\mathrm{G}\underline{\tilde{\mathbf{p}}_c^p}$$

$$\mathbf{u}_f^p = \underline{\Gamma_{sc}^{Vol}}\mathbf{u}_c^p$$

$$\underline{\mathrm{L}\tilde{\mathbf{p}}'_c} = \mathrm{M}\mathbf{u}_f^p$$

$$\mathbf{u}_f^{n+1} = \mathbf{u}_f^p - \underline{\mathrm{G}\tilde{\mathbf{p}}'_c}$$

$$\mathbf{u}_c^{n+1} = \mathbf{u}_c^p - \underline{\Gamma_{sc}^{Vol}}\underline{\mathrm{G}\tilde{\mathbf{p}}'_c} \qquad \underline{\tilde{\mathbf{p}}_c^{n+1} = \tilde{\mathbf{p}}_c^p + \tilde{\mathbf{p}}'_c}$$

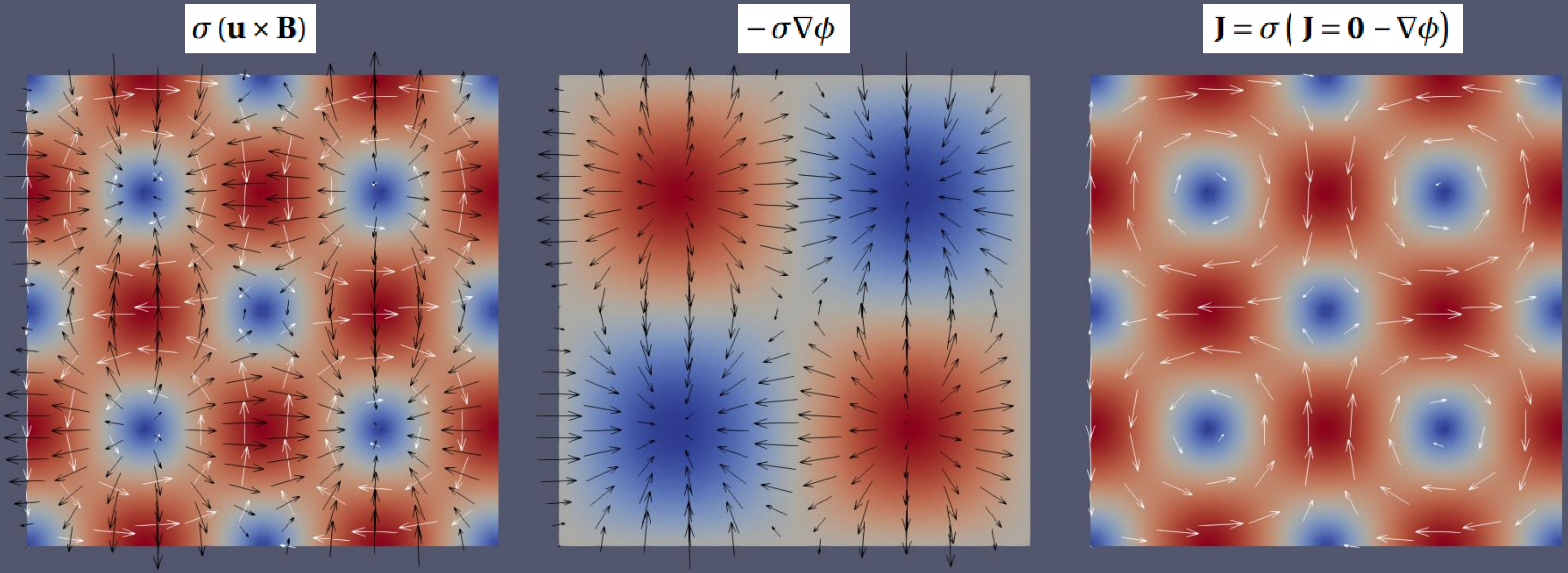
$$\mathbf{J}_c^p = \mathbf{u}_c^{n+1} \times \mathbf{B}_c^{n+1} - \underline{\Gamma_{sc}^{Vol}}\underline{\mathrm{G}\boldsymbol{\varphi}_c^p}$$

$$\mathbf{J}_f^p = \underline{\Gamma_{sc}^{Vol}}\mathbf{J}_c^p$$

$$\underline{\mathrm{L}\tilde{\boldsymbol{\varphi}}'_c} = \mathrm{M}\mathbf{u}_f^p$$

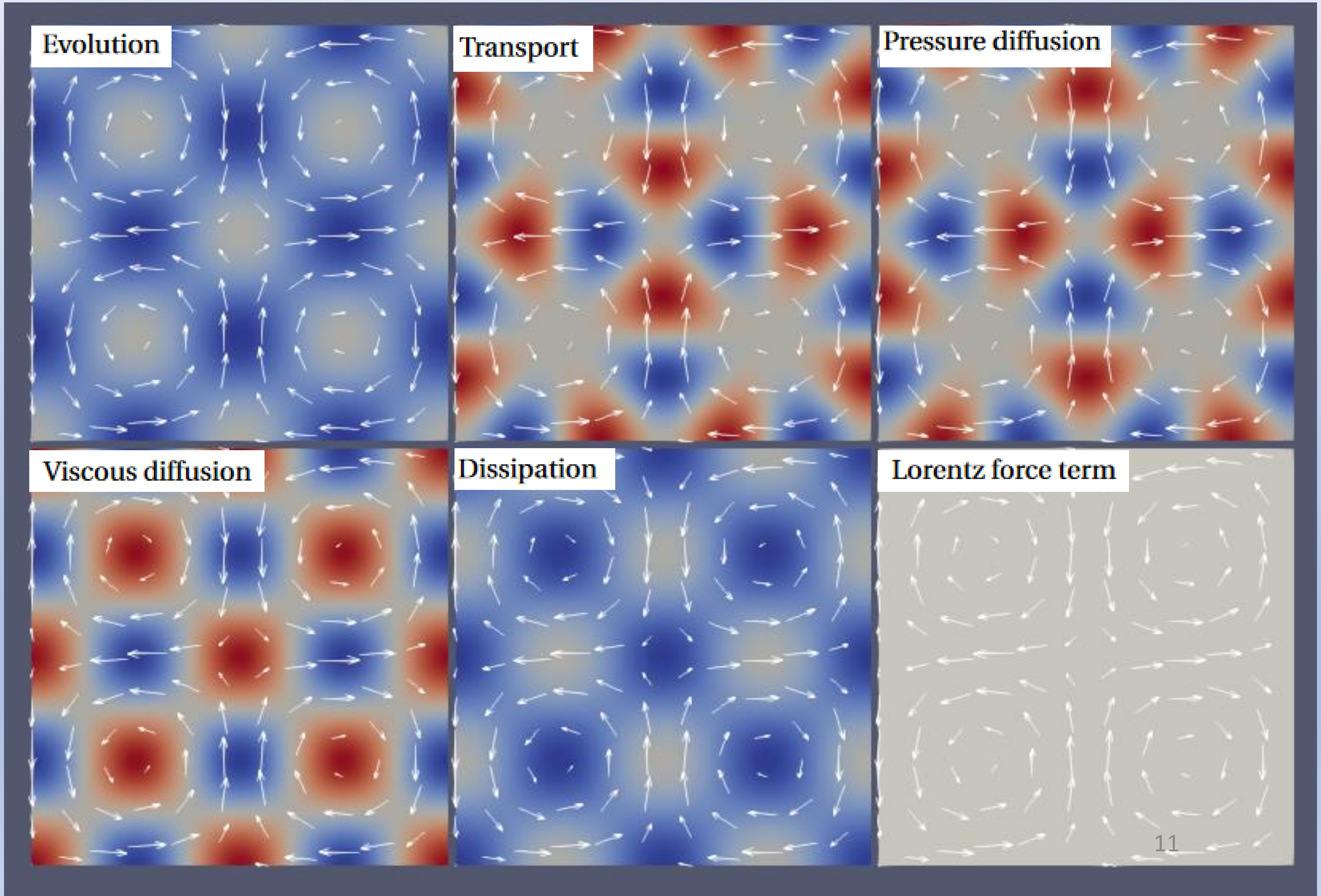
$$\mathbf{J}_c^{n+1} = \underline{\Gamma_{sc}^{Vol}}\left(\mathbf{J}_f^p - \mathrm{G}\underline{\tilde{\boldsymbol{\varphi}}'_c}\right) \qquad \underline{\boldsymbol{\varphi}_c^{n+1} = \boldsymbol{\varphi}_c^p + \tilde{\boldsymbol{\varphi}}'_c}$$

Taylor-Green vortex



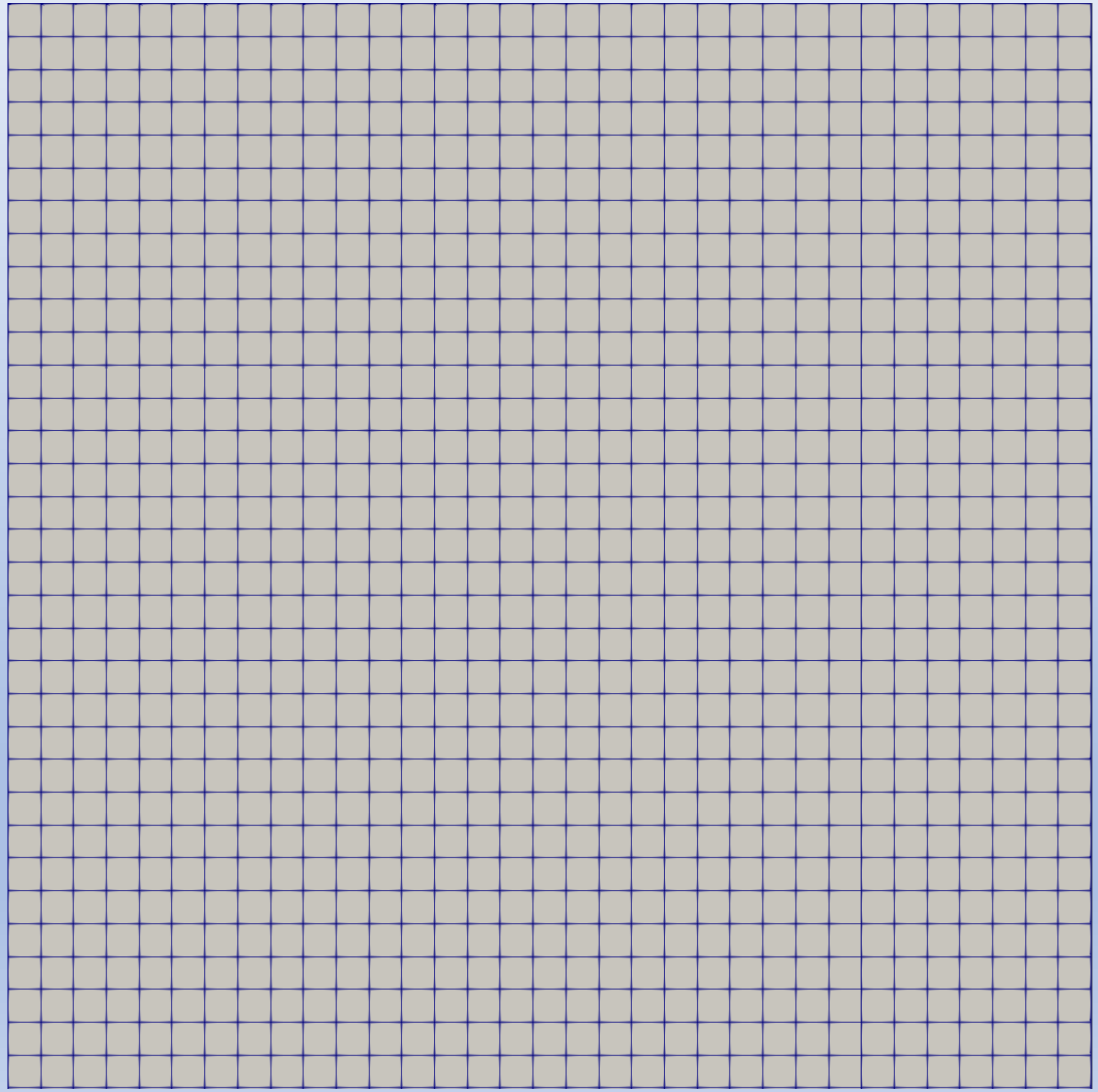
Taylor-Green vortex – Energy budgets

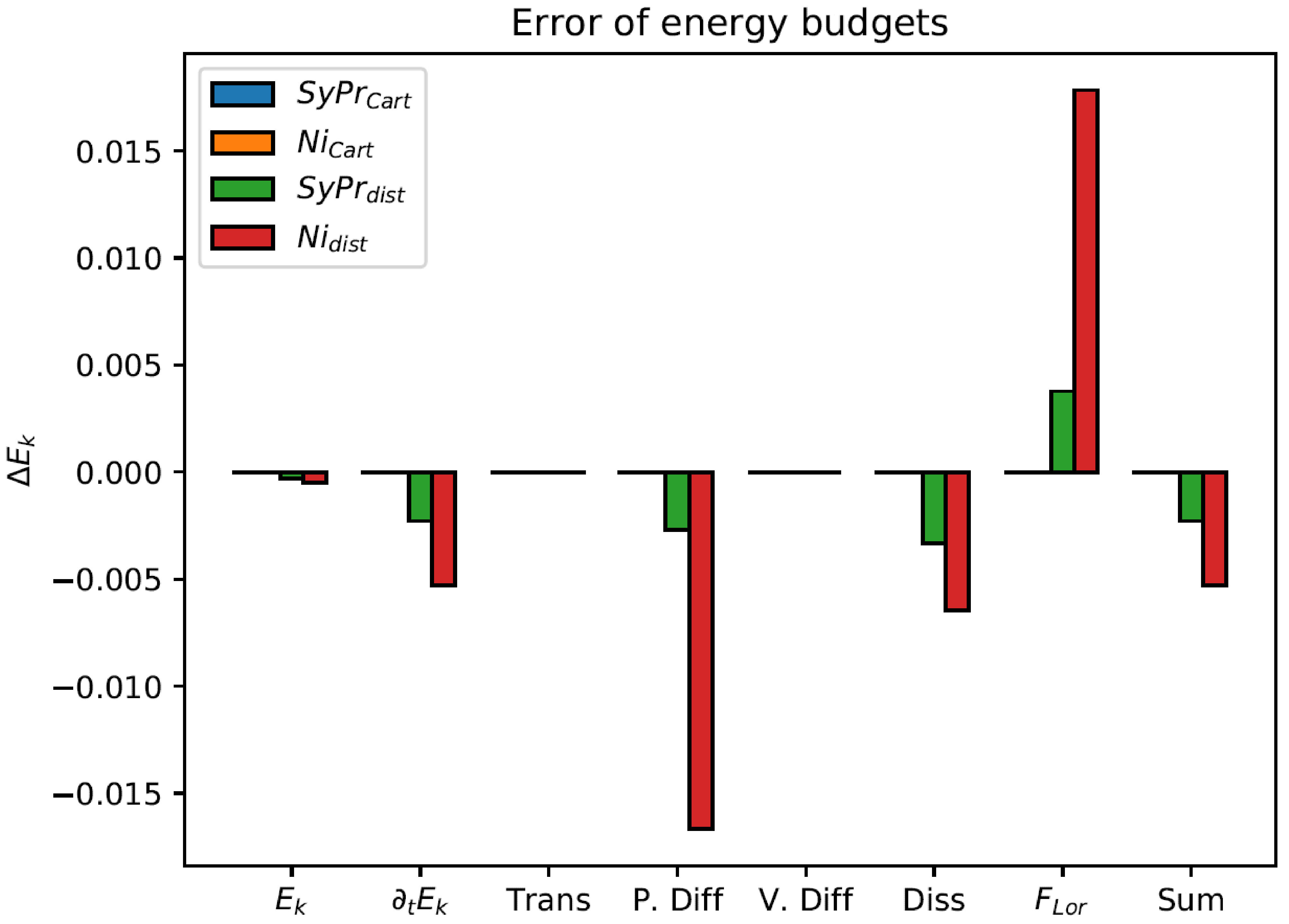
$$\mathbf{u} \cdot \left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \nu \nabla^2 \mathbf{u} - \nabla(\mathrm{p}/\rho) + \frac{\mathbf{J} \times \mathbf{B}}{\rho} \right)$$



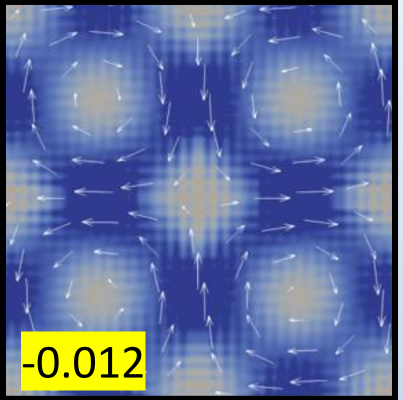
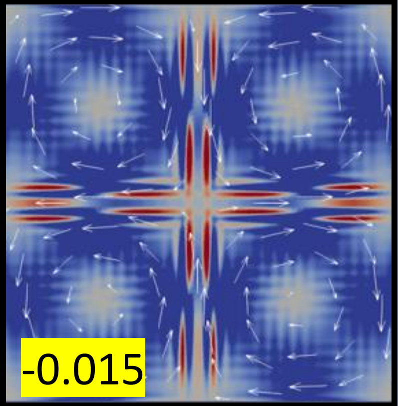
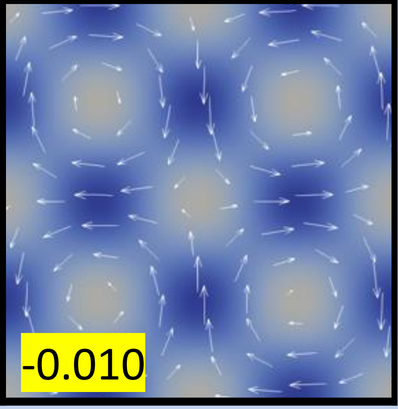
Taylor-Green vortex – grid distortion

Grid (65x65): Cartesian → distorted



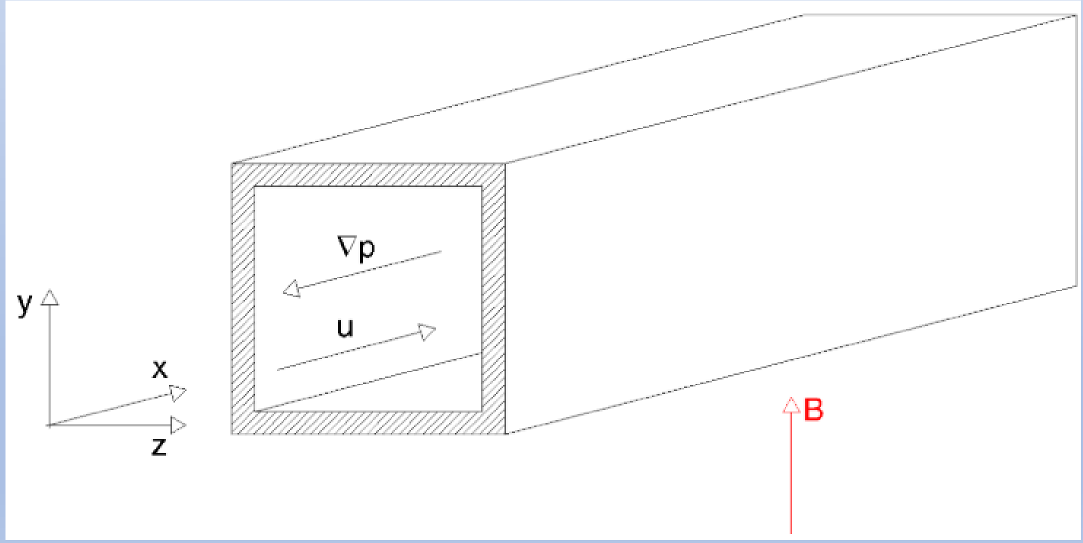


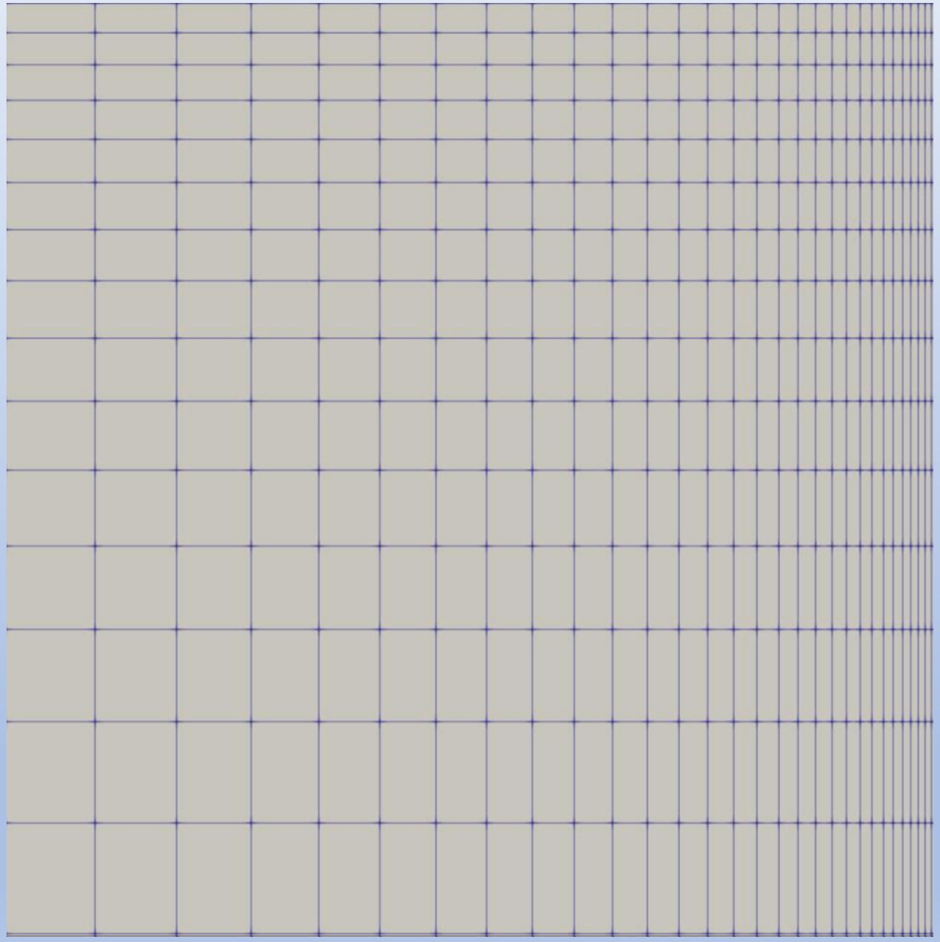
Error of energy budgets
$SyPr_{Cart}$
Ni_{Cart}
$SyPr_{dist}$
Ni_{dist}
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
ΔE_k
E_k
$\partial_t E_k$
Trans
P. Diff
V. Diff
Diss
F_{Lor}
Sum

	Evolution
SyPr	 -0.012
Ni	 -0.015
Ana	 -0.010

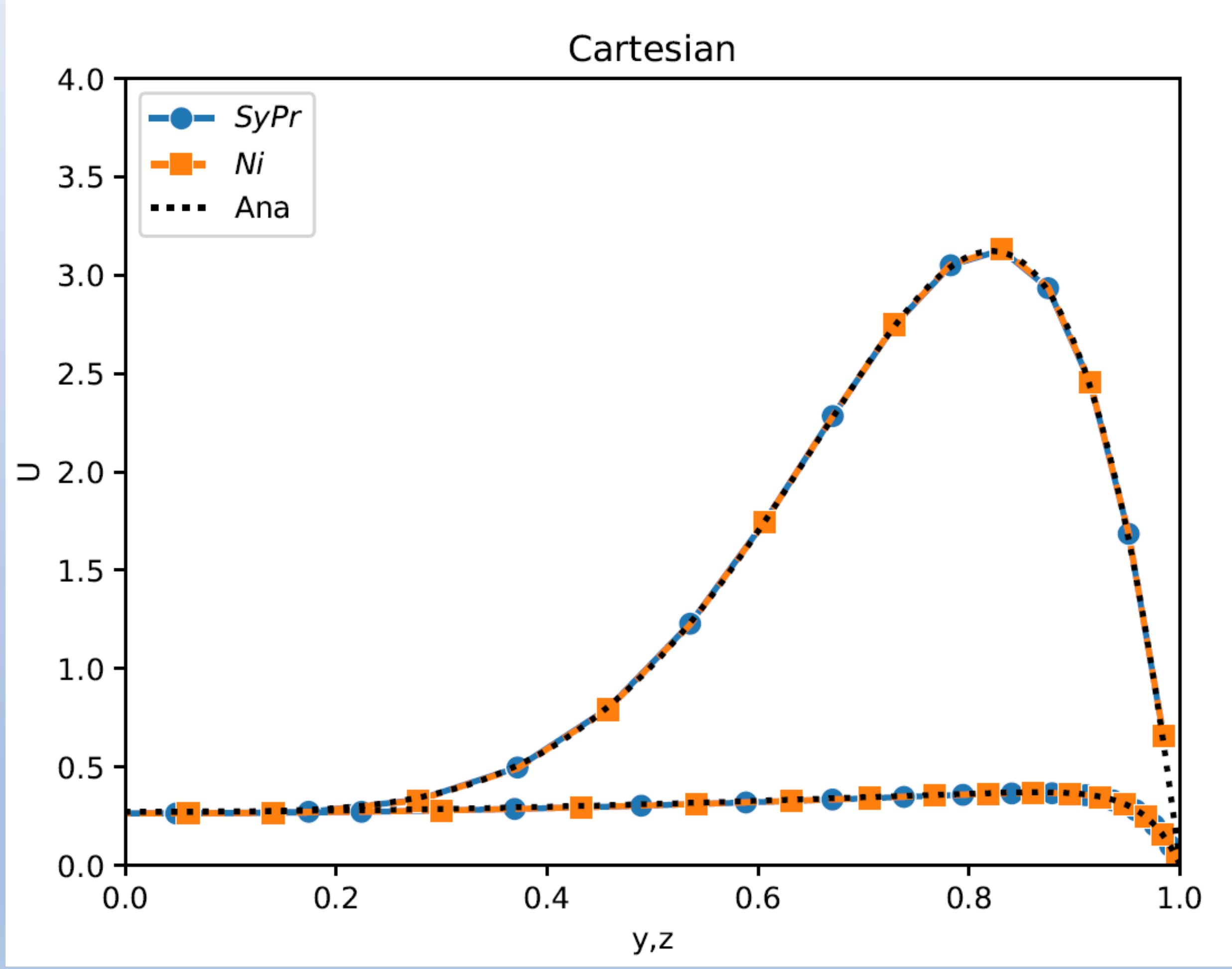
Hunt’s Case

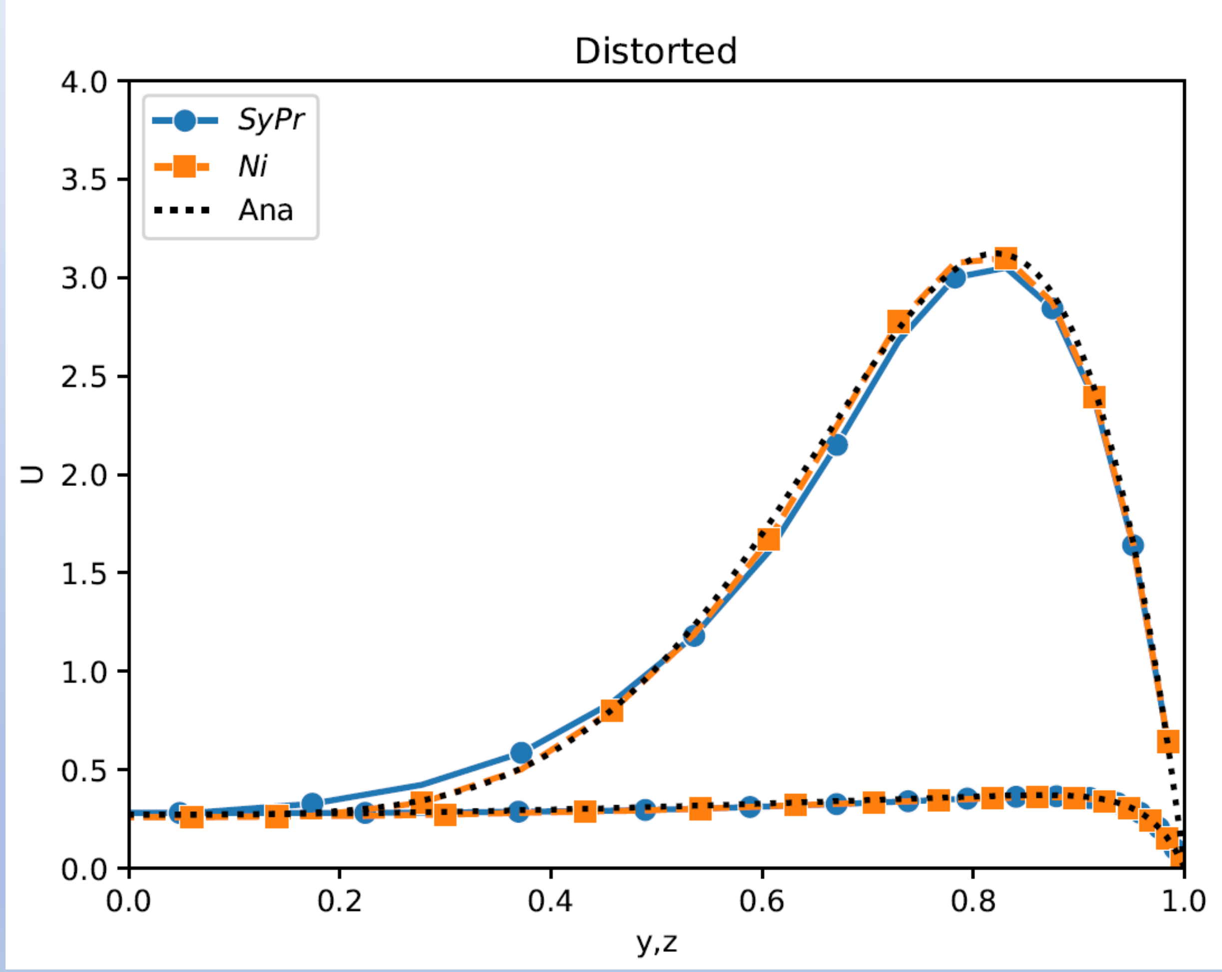
U_{mean} = 1.0
Ha = 30
Grid (65x65): Cartesian → distorted



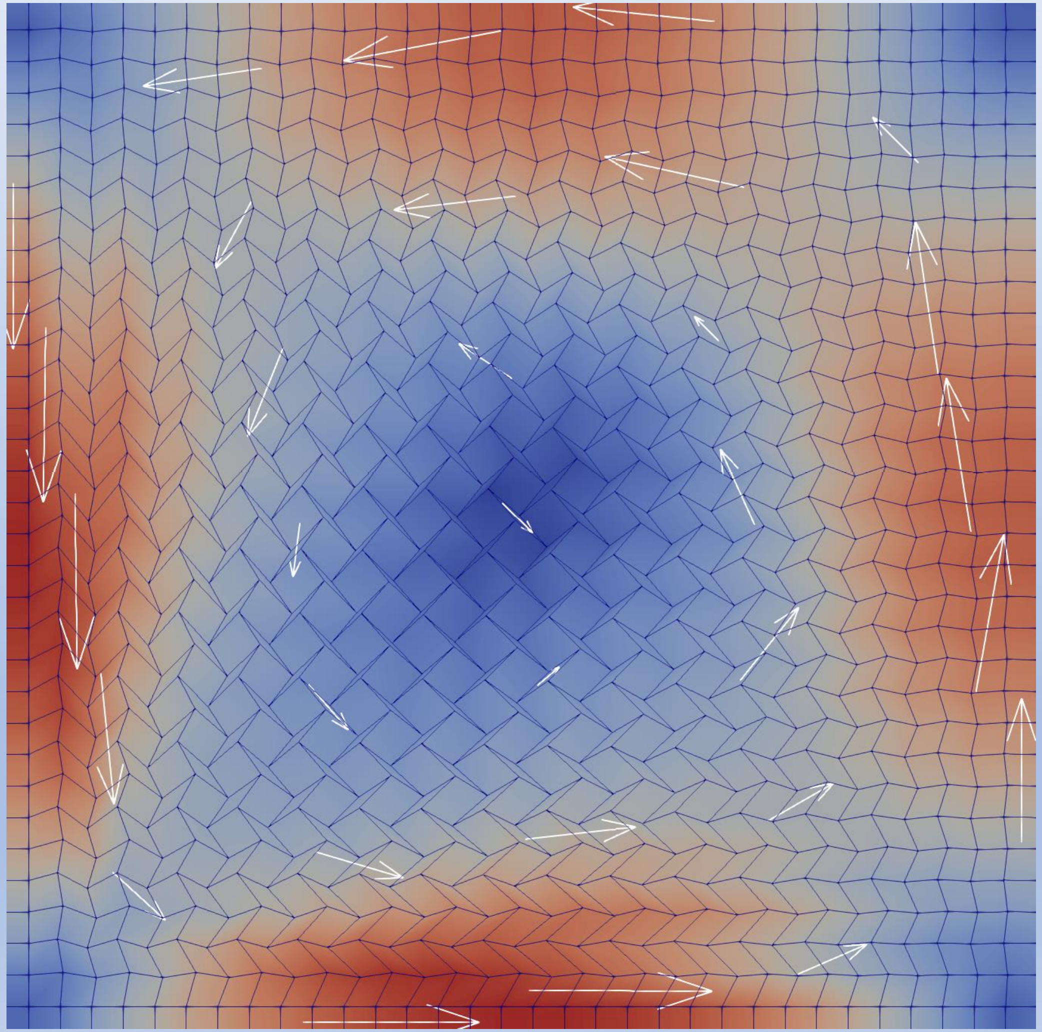


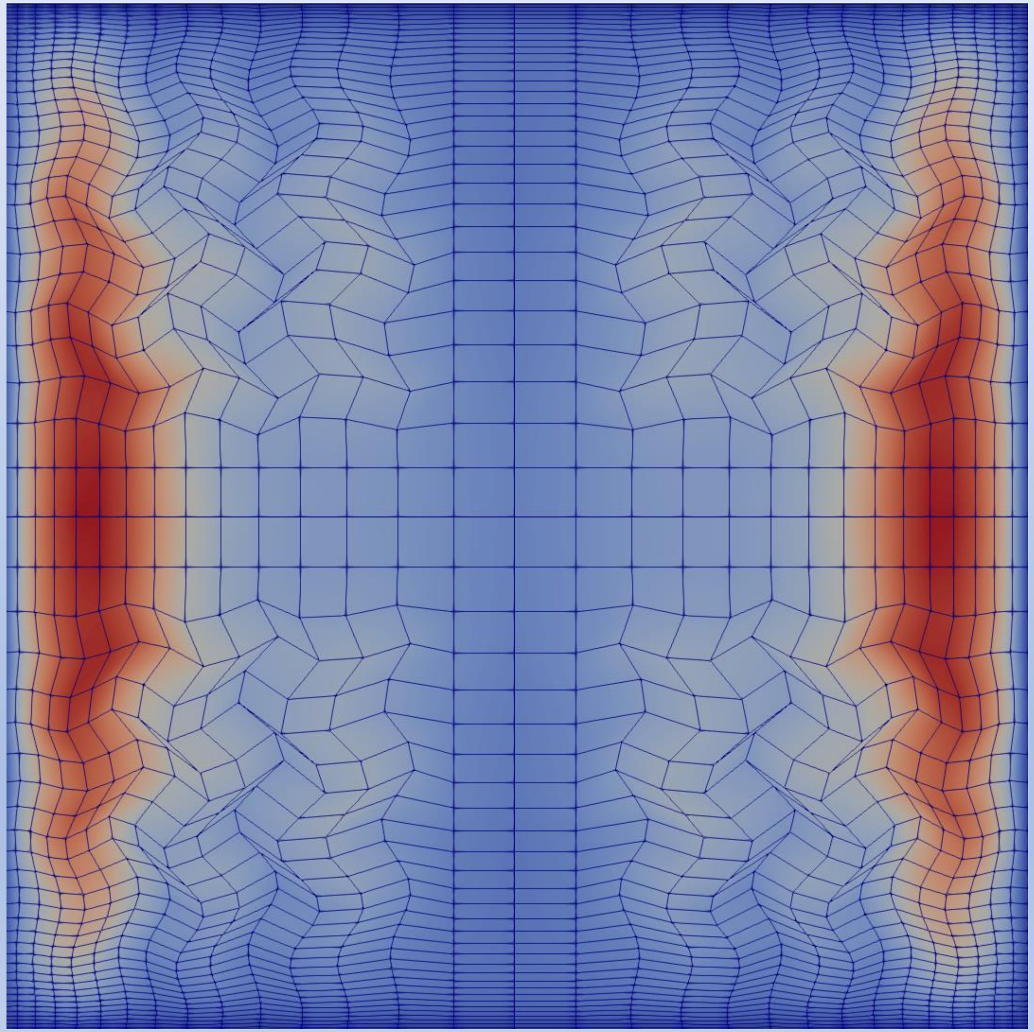
Hunt’s Case – Results





Unconditional stability!





Conclusions

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Thank you for your attention

Any questions?