ON THE CONDITIONS FOR A STABLE PROJECTION METHOD ON COLLOCATED UNSTRUCTURED GRIDS

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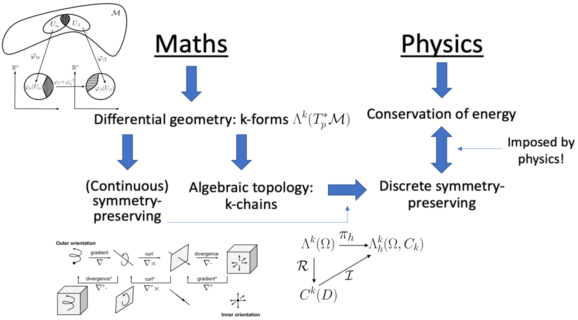
Motivation



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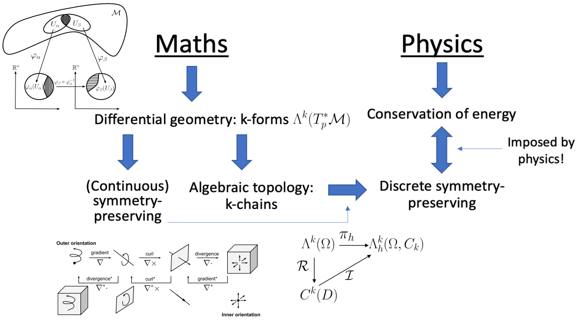

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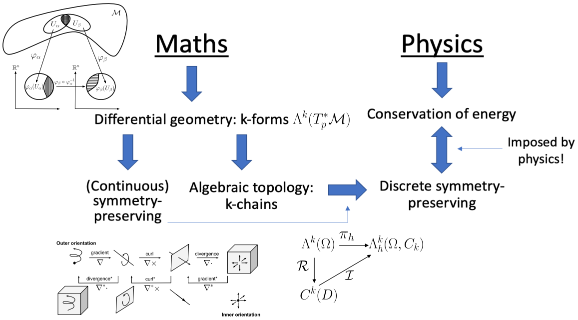

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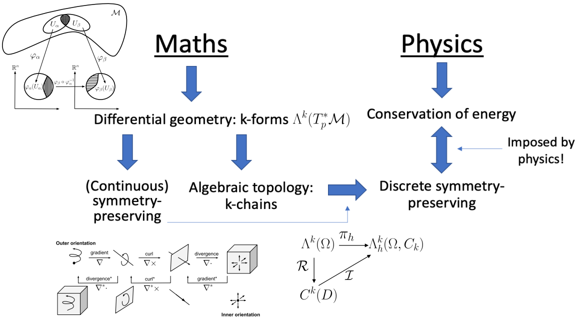

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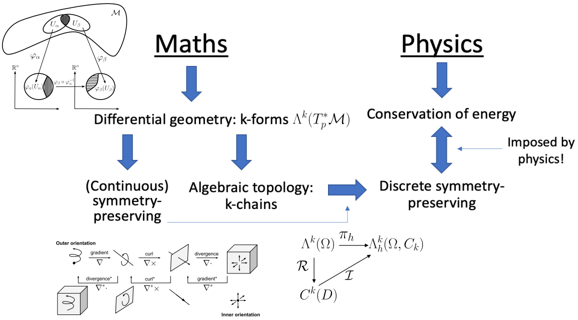

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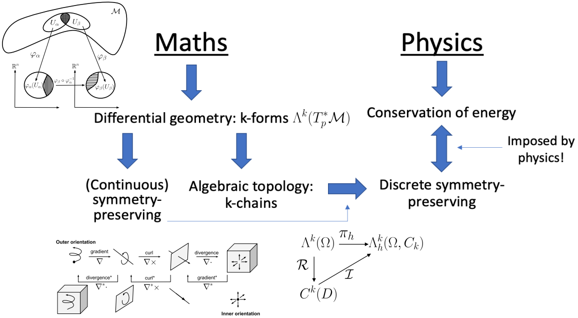

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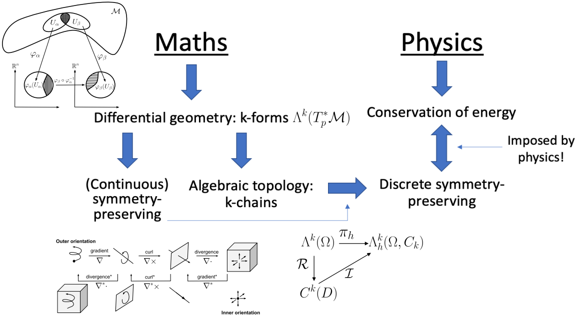

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  - Easily portable (to other codes, platforms...)
  - Unravel why using midpoint in the convective term.
  - Explain why the staggered metric is used at faces.

1. Definition of basic collocated operators

Let us suppose we have n control volumes and m faces.

Finite volume discretization of incompressible NS equations on an arbitrary collocated mesh

$$\Omega \frac{d\mathbf{u}_c}{dt} + C(\mathbf{u}_s)\mathbf{u}_c = -D\mathbf{u}_c - \Omega G_c \mathbf{p}_c, \quad (1)$$

$$M\mathbf{u}_s = \mathbf{0}_c. \quad (2)$$

- $\mathbf{p}_c = (p_1, ..., p_n)^T \in \mathbb{R}^n$ is the cell-centered pressure.
- $\mathbf{u}_c = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^T \in \mathbb{R}^{3n}$, where $\mathbf{u}_i = ((u_i)_1, ..., (u_i)_n)^T$ are the vectors containing the velocity components corresponding to the x_i-spatial direction.
- $\mathbf{u}_s = ((u_s)_1, ..., (u_s)_m)^T \in \mathbb{R}^m$ is the staggered velocity.
- The velocities are related via the interpolator from cells to faces $\Gamma_{c\to s} \in \mathbb{R}^{m\times 3n} \implies \mathbf{u}_s = \Gamma_{c\to s}\mathbf{u}_c$.

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Definition of basic collocated operators

The (3D) interpolator from cells to faces can be constructed as follows:

$$\Gamma_{c \to s} = N\Pi, \tag{3}$$

where

- $N = (N_{s,x} N_{s,y} N_{s,z}) \in \mathbb{R}^{m \times 3m}$ where $N_{s,x}, N_{s,y}, N_{s,z} \in \mathbb{R}^{m \times m}$ are diagonal matrices containing the x_i spatial component of the face normal vectors.
- $\Pi = I_3 \otimes \Pi_{c \to s} \in \mathbb{R}^{3m \times 3n}$.
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- $\Omega_c \in \mathbb{R}^{n\times n}$ is a diagonal matrix with the cell-centered volumes $\implies \quad \Omega = I_3 \otimes \Omega_c$.
- $C_c(\mathbf{u}_s) \in \mathbb{R}^{n\times n}$ is the cell-centered convective operator for a discrete scalar field $\implies \quad C(\mathbf{u}_s) = I_3 \otimes C_c(\mathbf{u}_s)$.
- $D_c \in \mathbb{R}^{n\times n}$ is the cell-centered diffusive operator for a discrete scalar field $\implies \quad D = I_3 \otimes D_c$.

Finally,

- $G_c \in \mathbb{R}^{3n\times n}$ represents the discrete collocated gradient.
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Mimicking continuous properties

Mimicking Hilbert adjointness in L^2 inner product $\rightarrow$

$$G = -\Omega_s^{-1} M^T,$$

Mimicking continuous Laplacian $\rightarrow$

$$L = MG = -M\Omega_s^{-1} M^T,$$

$$L_c = M_c G_c = -M\Gamma_{c\rightarrow s}\Omega^{-1}\Gamma_{c\rightarrow s}^T M^T,$$

Metric-consistency of L^2 inner product $\rightarrow$

$$\Gamma_{s\rightarrow c} = \Omega^{-1}\Gamma_{c\rightarrow s}^T \Omega_s. \quad (4)$$

where G is the center-to-face staggered gradient, L is the Laplacian operator, L_c is the collocated-Laplacian operator and $\Gamma_{s\rightarrow c}$ is the face-to-cell interpolator.

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Reduction of 2-forms and divergence operator

- Operator Reduction $\mathcal{R} \rightarrow$ FV discretization.
- 2-form $\mathbf{u_f}$ can be "understood" as a vector at the faces.

$$\mathcal{R}\mathbf{u}_f \equiv \int_f \mathbf{u}_f \cdot \hat{n} dS = \mathbf{u}_s S_f \tag{5}$$

Divergence operator:

$$\mathcal{R}(\nabla \cdot \mathbf{u}) = \int_V \nabla \cdot \mathbf{u} dV = \int_{\partial V} \mathbf{u}_f \cdot \hat{n} dS = \sum_f \mathbf{u}_s S_f \rightarrow$$

$$\rightarrow M = \mathbb{T}_{c \rightarrow s} S_f \rightarrow M\mathbf{u}_s = \delta \mathcal{R} \mathbf{u}_f \tag{6}$$

Skew-Symmetry of the convective operator

The convective operator in a FV context is constructed as:

$$C(\mathbf{u}_s)\mathbf{u}_c = M(diag(\mathbf{u}_s)\Pi^{\text{mid}}_{c\to s}\mathbf{u}_c) \qquad (7)$$

From a DEC point of view, it is possible to decompose the continuous convective for 3-forms as follows:

$$Conv(\phi) = di_{\mathbf{u}}\phi = d\star(\star\phi \wedge \mathbf{u}^\flat) \qquad (8)$$

Assuming a local stencil, the construction of the discrete wedge product between 0-forms and 1-forms induces the coefficient $1/2 \rightarrow$

It is not an interpolation, but the construction of the wedge.

Finally, this wedge combined with $M = \mathbb{T}_{c\to s}S_f$ is automatically skew-symmetric.

2. Conservation of global kinetic energy

Global kinetic energy equation

$$\frac{d||\mathbf{u}_c||^2}{dt} = -\mathbf{u}_c^T(C(\mathbf{u}_s) + C^T(\mathbf{u}_s))\mathbf{u}_c - \mathbf{u}_c^T(D + D^T)\mathbf{u}_c \\ -\mathbf{u}_c^T\Omega G_c\mathbf{p}_c - \mathbf{p}_c^T G_c^T\Omega^T\mathbf{u}_c. \quad (9)$$

In the absence of diffusion, that is, $D = 0$, the global kinetic energy is conserved if:

- $C(\mathbf{u}_s) = -C^T(\mathbf{u}_s)$, i.e, the convective operator should be skew-symmetric.
- $(-\Omega G_c)^T = M\Gamma_{c\to s}$, because $M\mathbf{u}_s = \mathbf{0}_c$.

Conservation of global kinetic energy

Global kinetic energy equation with skew-symmetric convective operator

$$\frac{d||\mathbf{u}_c||^2}{dt} = -\mathbf{u}_c^T(D + D^T)\mathbf{u}_c - \mathbf{u}_c^T \Omega G_c \mathbf{p}_c - \mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c.$$

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- $(-\Omega G_c)^T = M\Gamma_{c\rightarrow s}$, because $M\mathbf{u}_s = \mathbf{0}_c$ (But this relation is exact ONLY in staggered configurations!).

In collocated framework, we either solve:

$$M\mathbf{u}_s = 0 \rightarrow Lp_c = M\Gamma_{c\rightarrow s}\mathbf{u}_c^p \rightarrow \text{Kinetic Energy Error} \quad (10)$$

$$M_c\mathbf{u}_c = 0 \rightarrow L_c p_c = M\Gamma_{c\rightarrow s}\mathbf{u}_c^p \rightarrow \text{Checkerboard} \quad (11)$$

Conservation of global kinetic energy

Global kinetic energy equation with skew-symmetric convective operator

$$\frac{d||\mathbf{u}_c||^2}{dt} = -\mathbf{u}_c^T(D + D^T)\mathbf{u}_c - \mathbf{u}_c^T \Omega G_c \mathbf{p}_c - \mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c.$$

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In collocated framework and explicit time integration, the (artificial) kinetic energy added is given by:

$$-\mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c = \mathbf{p}_c^T (L - L_c)\mathbf{p}_c \Delta t \qquad (12)$$

Volume-weighted interpolator

- The volume-weighted interpolator can be constructed as:

$$\Pi_{c\to s} = \Delta_s^{-1}\Delta_{sc}^T, \tag{13}$$

where $\Delta_s \in \mathbb{R}^{m\times m}$ is a diagonal matrix containing the projected distances between two adjacent control volumes, and $\Delta_{sc} \in \mathbb{R}^{m\times n}$ contains the projected distances between an adjacent cell node and its corresponding face.

Volume-weighted interpolation: $\phi_f = \frac{\tilde{V}_{1,f}}{\tilde{V}_{1,f}+\tilde{V}_{2,f}}\phi_{c1} + \frac{\tilde{V}_{2,f}}{\tilde{V}_{1,f}+\tilde{V}_{2,f}}\phi_{c2}$.

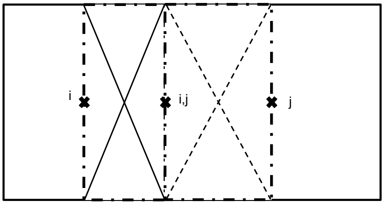

Figure 1: Volume-weighted volumes

Volume-weighted interpolator

Volume-weighted interpolation: $\phi_f = \frac{d_{1,f}}{d_{1,f}+d_{2,f}}\phi_{c1} + \frac{d_{2,f}}{d_{1,f}+d_{2,f}}\phi_{c2}$.

Momentum is conserved when interpolated by the volume-weighted:

$$(\mathbf{u}_c, \mathbf{1}_c)_\Omega = (\mathbf{u}_s, \mathbf{1}_s)_{\Omega_s} \rightarrow$$

$$\mathbf{u}_x^1 \tilde{V}_1 + \mathbf{u}_x^2 \tilde{V}_2 = (a_1 \mathbf{u}_x^1 + a_2 \mathbf{u}_x^2) A_f = (d_1 \mathbf{u}_x^1 + d_2 \mathbf{u}_x^2) A_f \rightarrow$$

$$\mathbf{u}_x^1 \tilde{V}_1 + \mathbf{u}_x^2 \tilde{V}_2 = (\frac{d_1}{d_1 + d_2}\mathbf{u}_x^1 + \frac{d_2}{d_1 + d_2}\mathbf{u}_x^2) V_s$$

And the staggered metric is induced automatically to the faces!

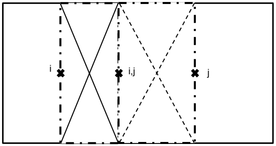

Figure 2: Volume-weighted volumes

Conservation of global kinetic energy

Stable solutions → Eigenvalues of $L - L_c$ negative.

Theorem

Assumptions:

Conservation of global kinetic energy

Stable solutions $\rightarrow$ Eigenvalues of $L - L_c$ negative.

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- Our projection method adds a kinetic energy error of the form $\mathbf{p}_c^T(L - L_c)\mathbf{p}_c^T$ (such as the FSM or PISO).

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Then, $\mathbf{p}_c^T(L - L_c)\mathbf{p}_c^T \leq 0$ at each time step $\iff$

1. The volume-weighted cell-to-face interpolator is used for the pressure gradient interpolator.

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- Our projection method adds a kinetic energy error of the form $\mathbf{p}_c^T(L - L_c)\mathbf{p}_c^T$ (such as the FSM or PISO).
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Then, $\mathbf{p}_c^T(L - L_c)\mathbf{p}_c^T \leq 0$ at each time step $\iff$

1. The volume-weighted cell-to-face interpolator is used for the pressure gradient interpolator.
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Stable solutions $\rightarrow$ Eigenvalues of $L - L_c$ negative.

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Corollary 1

Hexahedral (cuboid) meshes always give stable results when using the volume-weighted interpolator.

Corollary 2

Triangular meshes give stable results when using the volume-weighted interpolator if the node is placed at the circumcenter.

Numerical robustness

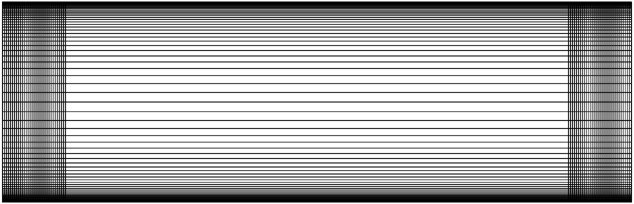


Figure 3: Highly distorted mesh in a $Re_\tau = 395$ channel flow. Max aspect ratio is 250.

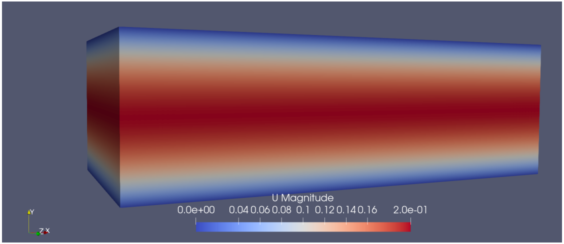
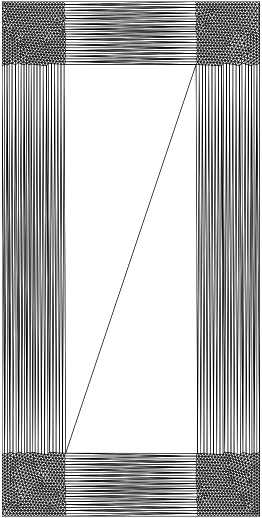


Figure 4: Velocity converged with the volume-weighted interpolator.

Numerical robustness



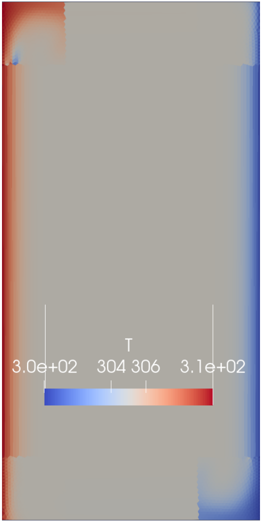


Figure 5: Test of the method's robustness in a $Ra = 10^6$ differentially heated cavity.

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Ongoing work

- Find the conditions for tetrahedral meshes in order to satisfy the geometrical conditions of the theorem.