

Stability-aware Time Integration for Reduced Order Models for Incompressible Flows

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Introduction

Ultimate goal of Reduced Order Modeling

- Reduce the computational time to the solution of a problem.

Reference	Speed-up
Baiges et al., <i>Int. J. Numer. Meth. Fluids</i> 72 (2013)	3x - 434x (*)
Reyes and Codina, <i>Comput. Methods Appl. Mech. Eng.</i> 363 (2020)	7x - 13x (*)
Rosenberger and Sanderse, <i>J. Comput. Phys.</i> 491 (2023)	100x
Klein and Sanderse, <i>J. Comput. Phys.</i> 499 (2024)	600x (*)

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What do they have in common?

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What do they have in common?

- The **same** time step size is used for the FOM and the ROM.

Farhat et al. (2017)¹ and Baiges et al. (2013) observed that ROM can be stable for time step sizes that are unstable for the FOM.

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Research question #2:

Can we quantify the maximum time step size for which a ROM is stable in such a way that it can be used to design a stability-aware time integration strategy?

- The method should be applicable to any ROM for incompressible flows, regardless of the time integration scheme used.
- The operations used to compute the maximum time step size should be ROM-size, i.e., they should not depend on the size of the FOM.
- The method should be generalizable to hyperreduced ROMs. Therefore, online operations should not exceed $\mathcal{O}(M)$.

Stability-aware time integration for FOMs

$$M\mathbf{u}_h(t) = \mathbf{g}_h(t)$$

$$\Omega \frac{d\mathbf{u}_h(t)}{dt} + C(\mathbf{u}_h(t))\mathbf{u}_h(t) = -\Omega G\mathbf{p}_h(t) + D\mathbf{u}_h(t) + \Omega\mathbf{f}_h(t)$$

EigenCD

(Trias and Lehmkuhl, *Numer. Heat Transfer: B* **60** (2011))

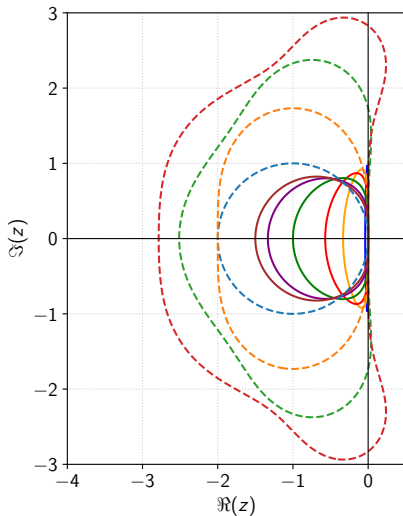
- 1 Reconstruct the explicit operator $D, C(\mathbf{u}_h(t))$.
- 2 Bound the eigenvalues of the explicit operator with Gershgorin's theorem.
- 3 Compute the maximum time step size for which the time integration scheme is stable.

AlgEigCD

(Trias et al., *Comput. Phys. Commun.* **305** (2024))

- 1 Using the structure of the *collocated* operators, precompute a matrix that allows not to reconstruct the explicit operator $D, C(\mathbf{u}_h(t))$.
- 2 Bound the eigenvalues of the explicit operators with a single SpMV.
- 3 Compute the maximum time step size for which the time integration scheme is stable.

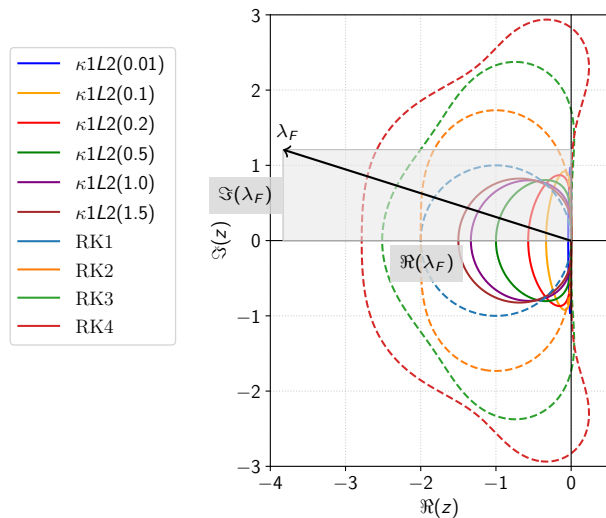
Stability-aware time integration for FOMs



Stability-aware time integration for FOMs

- Compute λ_F (with whichever method)

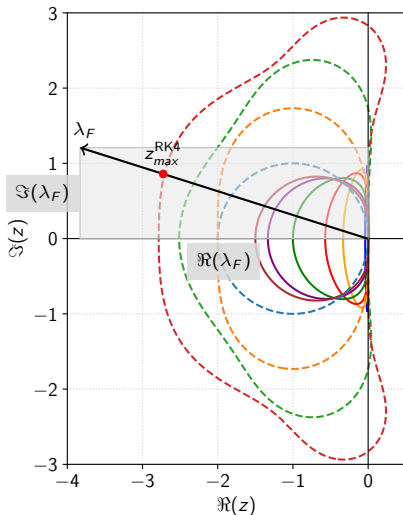
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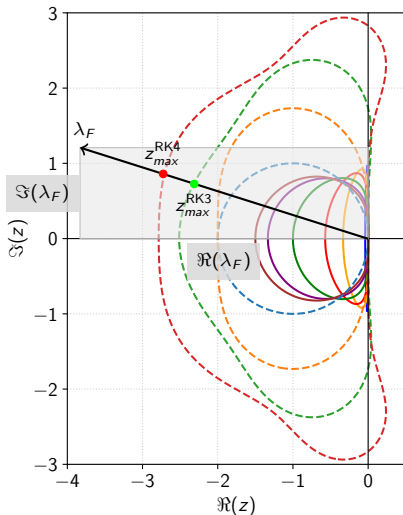


Stability-aware time integration for FOMs

- Compute λ_F (with whichever method)
- Compute Δt_{max} for which the time integration scheme is stable.

$$\Delta t = z_{max} / \|\lambda_F\|$$

Stability-aware time integration for FOMs

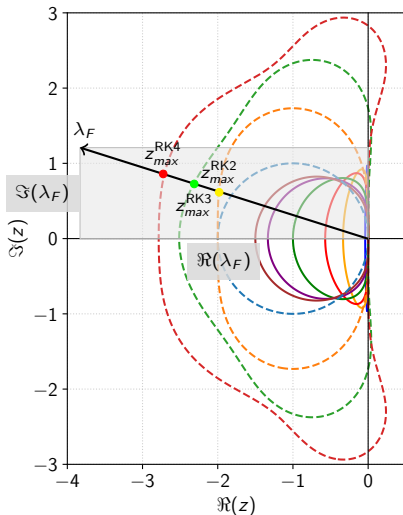


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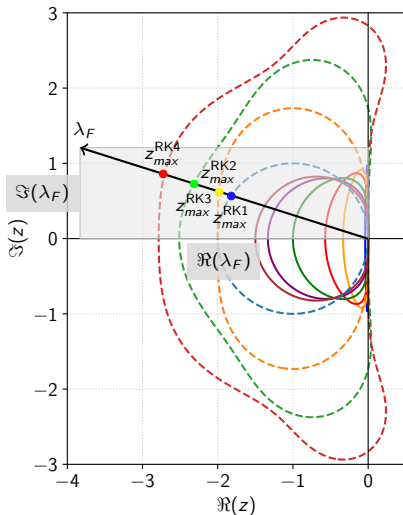


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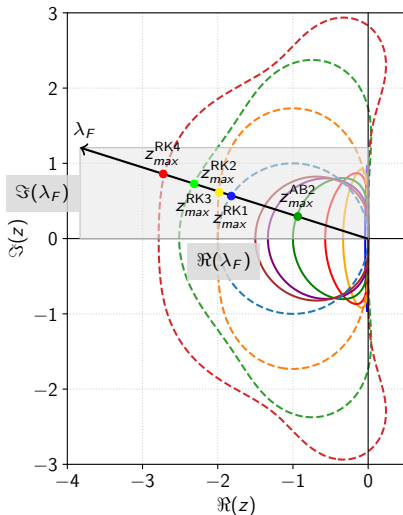


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Energy-conserving POD-Galerkin ROMs for incompressible flows

Following the ansatz $\mathbf{u}_h(t) \approx \Phi \mathbf{a}(t)$, the POD-Galerkin ROM for incompressible flows reads:

POD-Galerkin ROMs for incompressible flows

$$\frac{d\mathbf{a}}{dt} = -C_r(\mathbf{a} \otimes \mathbf{a}_c) - C_{r,bc}(\mathbf{a} \otimes \mathbf{a}_{bc}) - C_{bc}(\mathbf{a}_{bc} \otimes \mathbf{a}_{bc}) + D_r \mathbf{a} + \mathbf{f}_r$$

- $C_{r,i} = \Phi^T C(\Phi_i) \Phi$: skew-symmetric, precomputed offline $\rightarrow$ extends offline computational time, but eliminates FOM-size operations in the online stage ($\mathcal{O}(M^3)$!!!! only suitable for low M).

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$$\frac{d\mathbf{a}}{dt} = - \sum_{i=1}^M a_{c,i} C_{r,i} \mathbf{a} - \sum_{i=1}^{M_{bc}} a_{bc,i} C_{r,bc,i} \mathbf{a} - \sum_{i=1}^{M_{bc}} a_{bc,i} C_{bc,i} \mathbf{a}_{bc} + D_r \mathbf{a} + \mathbf{f}_r$$

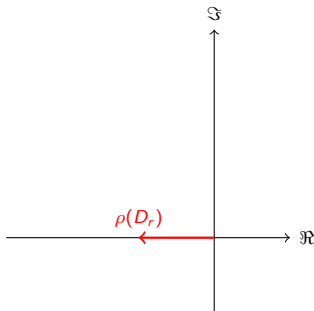
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Stability-aware time integration for ROMs

Recycling the FOM stability-aware time integration methods for ROMs

- EigenCD: does not require any modification, it is based on the explicit (reduced) operator. $\mathcal{O}(M^2)$ operations.
- AlgEigCD: cannot be used $\rightarrow$ it is based on the structure of the *collocated* operators, which is not preserved in the ROM, and ends up being FOM-size. $\mathcal{O}(N)$ operations.

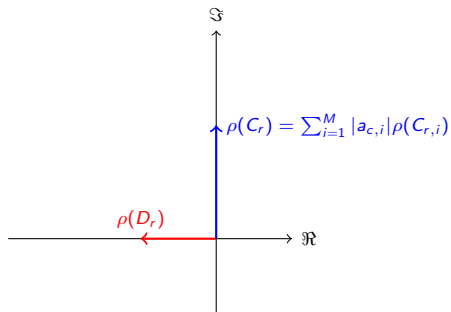
RedEigCD: a stability-aware time integration method for ROMs



Diffusive operator's eigenbound

Computed **exactly**: offline
computation of $\mathcal{O}(M^3)$

RedEigCD: a stability-aware time integration method for ROMs

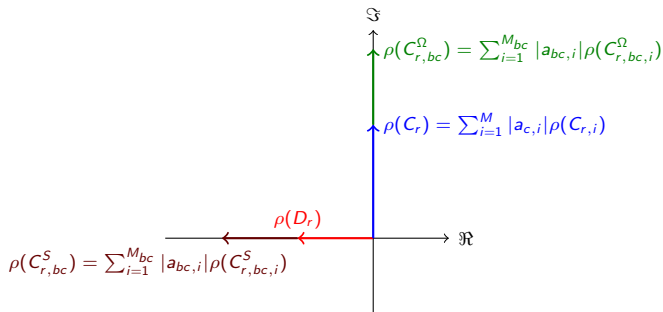


Convective operator's eigenbound

$\rho(C_{r,i})$ is computed **exactly** offline.
 $\rho(C_r)$ as a weighted sum.

- $\mathcal{O}(M)$ online operations.

RedEigCD: a stability-aware time integration method for ROMs



Splitting of the boundary condition term

No guarantee on symmetry/skew-symmetry of $C_{r,bc}$.

$$C_{r,bc}^S = \frac{1}{2}(C_{r,bc} + C_{r,bc}^T)$$

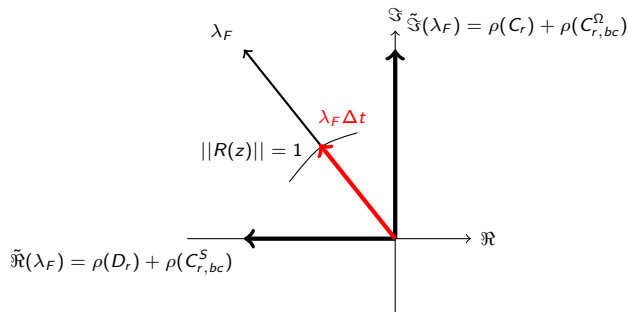
$$C_{r,bc}^\Omega = \frac{1}{2}(C_{r,bc} - C_{r,bc}^T)$$

$\rho(C_{r,bc,i}^X)$ is computed **exactly** offline.

$\rho(C_{r,bc}^X)$ as a weighted sum.

- $\mathcal{O}(M_{bc})$ online operations.

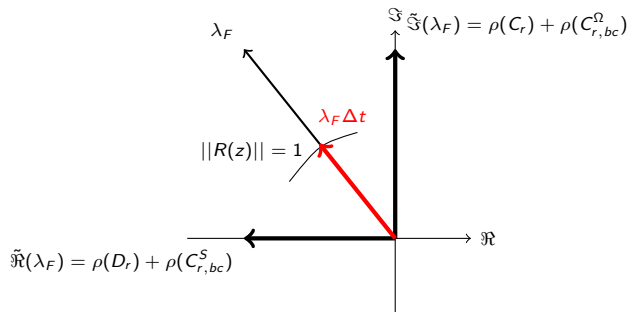
RedEigCD: a stability-aware time integration method for ROMs



Determining the maximum stable Δt

- 1 Compute $\Re(\lambda_F)$ and $\Im(\lambda_F)$.
- 2 Compute z_{max} with the bisection method.
- 3 Set $\Delta t = z_{max} / \|\lambda_F\|$.

RedEigCD: a stability-aware time integration method for ROMs



Determining the maximum stable Δt

- 1 Compute $\Re(\lambda_F)$ and $\Im(\lambda_F)$.
- 2 Compute z_{max} with the bisection method.
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Research question #2:

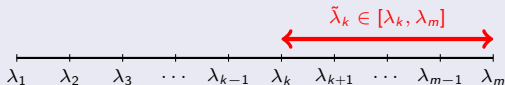
Can we quantify the maximum time step size for which a ROM is stable in such a way that it can be used to design a stability-aware time integration strategy?

Analytical stability analysis of ROMs

Diffusive operator

Bach et al.⁴ used Poincaré separation theorem to prove that the eigenvalues of $V^T A V$ are bounded by the eigenvalues of A if A is symmetric and V is orthogonal

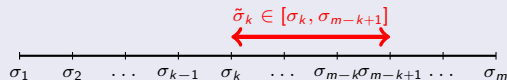
$$\tilde{\lambda}_k \in [\lambda_k, \lambda_m] \Rightarrow \tilde{\lambda}_k \leq \lambda_m$$



Convective operator

Rao⁵ extended the PST to singular values, which holds for any matrix A and V orthogonal,

$$\tilde{\sigma}_k \in [\sigma_k, \sigma_{m-k+1}] \Rightarrow \tilde{\sigma}_k \leq \sigma_m \Rightarrow \tilde{\lambda}_k \leq \lambda_m$$



⁴Bach et al. *ArXiv* (1806.11404) [preprint]

⁵Rao, *J. Multivar. Anal.* **9** (1979)

Analytical stability analysis of ROMs

Research question #1:

Can the statement above be proven in a general way for ROMs for incompressible flows?

Theorem

Let Δt^{ROM} be the largest stable timestep for the ROM given a state $\mathbf{a}$, and Δt^{FOM} the largest stable timestep for a state $\mathbf{u}$ that is in the span of Φ , e.g. $\mathbf{u} = \Phi \mathbf{a}$, both using the same explicit time integration scheme. Then, $\Delta t^{\text{ROM}} \geq \Delta t^{\text{FOM}}$.

- Only strictly valid if $\mathbf{u} = \Phi \mathbf{a}$: serves as a theoretical proxy for the stability of the ROM.

Numerical experiments

Test cases

- Shear-layer roll-up at $Re=1000$
- Unsteady wake behind an actuator disk at $Re=100$
- Two-dimensional simulations
- $M = [16, 32, 64, 128, 200]$ modes
- Integrated using a classical RK4 scheme

Case	Method
A	FOM+AlgEigCD
B	ROM+RedEigCD
C	ROM+constant Δt
D	$\mathbf{a}_{\text{best}} = \Phi^T \Omega \mathbf{u}$

Outline of the numerical experiments

- Evaluate $\Delta t^{\text{ROM}} / \Delta t^{\text{FOM}}$
- Evaluate the error of the ROM with respect to the best-approximation ROM
- Compare the eigenbound estimation error with EigenCD and RedEigCD
- Evaluate the computational performance of RedEigCD with respect to the FOM.

Numerical experiments

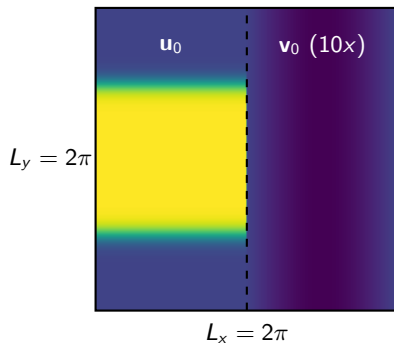
Shear-layer roll-up at $Re=1000$

- Initial conditions:

$$u_0(x, y) = 1 + \begin{cases} \tanh\left(\frac{y-\pi/2}{\delta}\right), & y \leq \pi, \\ \tanh\left(\frac{3\pi/2-y}{\delta}\right), & y > \pi, \end{cases}$$

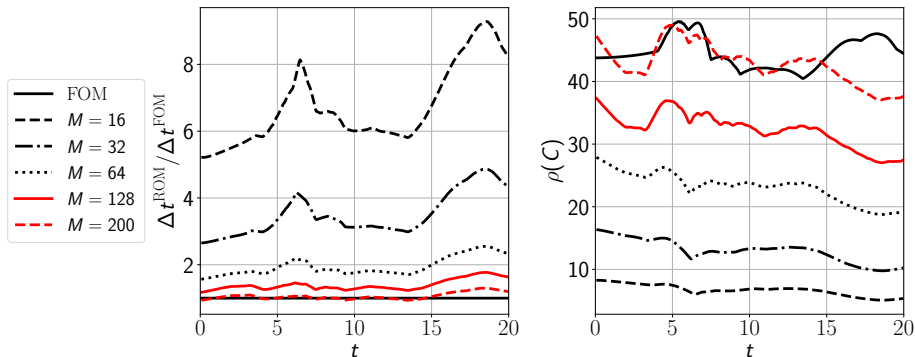
$$v_0(x, y) = \epsilon \sin(x)$$

- $\delta = \pi/15$, $\epsilon = 1/20$
- Fully periodic domain
- FOM configuration (Case A):
 - $N_x = N_y = 100$ cells
 - RK4 time integration scheme
 - Δt determined with AlgEigCD



Numerical experiments

Shear-layer roll-up at $Re=1000$

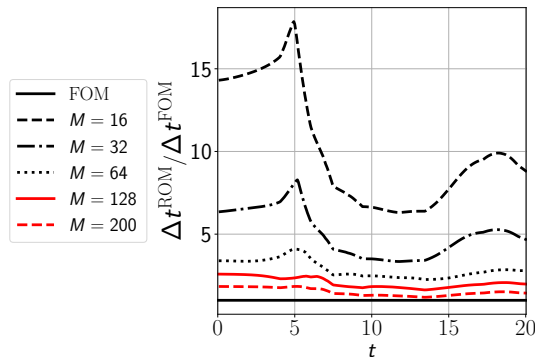


Observations

- Δt can be up to 9x larger for the ROM than for the FOM.
- For high values of M , $\Delta t^{\text{ROM}} \approx \Delta t^{\text{FOM}}$, or even $\Delta t^{\text{ROM}} < \Delta t^{\text{FOM}}$.

Numerical experiments

Shear-layer roll-up at $Re=1000$

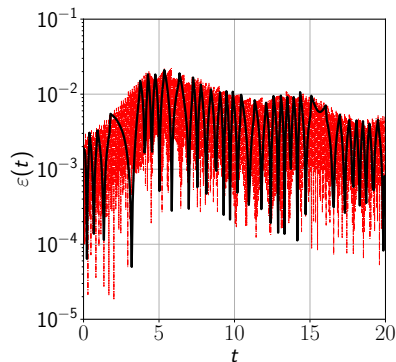
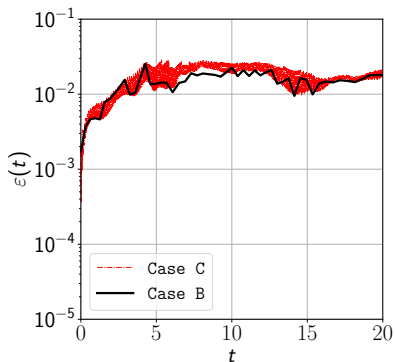


Observations

- With the best ROM (Case D), $\Delta t^{\text{ROM}} > \Delta t^{\text{FOM}}$, as expected from the analytical stability analysis.

Numerical experiments

Shear-layer roll-up at $Re=1000$



Numerical experiments

Unsteady wake behind an actuator disk at $Re=100$

- Initial conditions:

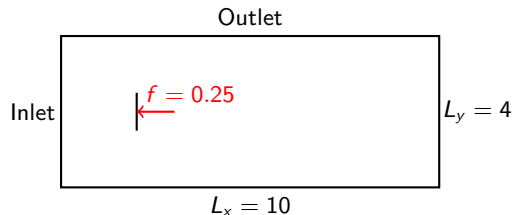
$$u_0(x, y) = 1, \quad v_0(x, y) = 0$$

- Inlet boundary condition:

$$u(0, y) = \cos(\alpha(t)), \quad v(0, y) = \sin(\alpha(t))$$

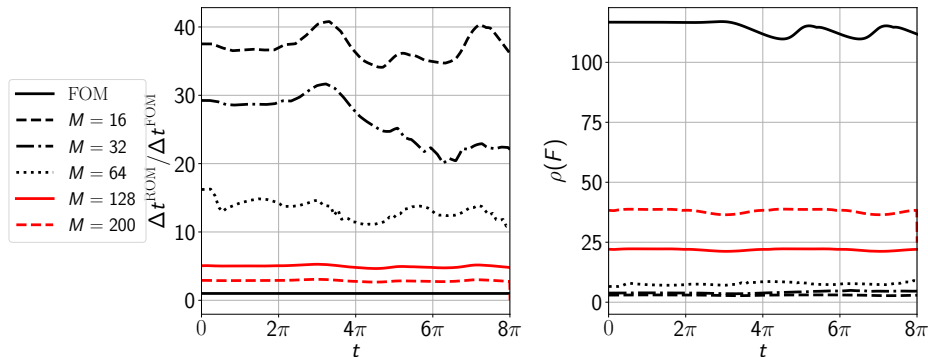
$$\alpha(t) = \frac{\pi}{6} \sin(t/2)$$

- Momentum sink at $(2, 0)$ to model the actuator disk $f = 0.25$.
- FOM configuration (Case A):
 - 200 × 80 cells
 - RK4 time integration scheme
 - Δt determined with AlgEigCD



Numerical experiments

Unsteady wake behind an actuator disk at $Re=100$

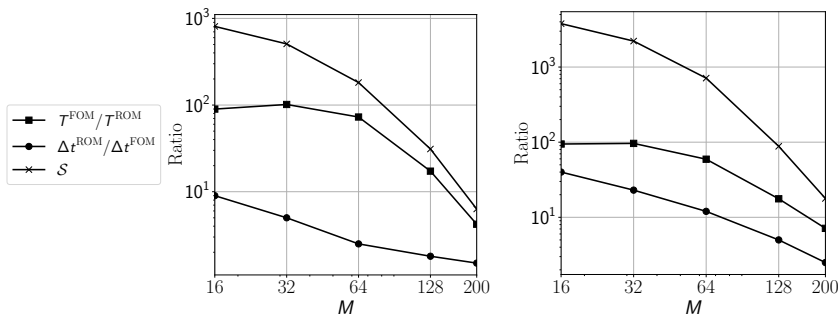


Observations

- Δt can be up to 40x larger for the ROM than for the FOM.
- $\Delta t^{\text{ROM}} > \Delta t^{\text{FOM}}$, regardless of the number of modes used.

Numerical experiments

Total speed-ups



Observations

- Speed-ups may reach up to 800x for the shear-layer roll-up case, and up to 2000x for the actuator disk case.
- Performance deprecates for $M > 64$ due to the $\mathcal{O}(M^3)$ complexity. Hyperreduction is required to reduce the online computational cost.

Conclusions and future work

Conclusions

- RQ#1 is solved: the maximum stable time step size for a ROM is always larger than or equal to the maximum stable time step size for the FOM.
- RQ#2 is solved: the method can be used to design a stability-aware time integration strategy for ROMs.
- The method yields speed-ups of up to 40x compared to ROMs with a constant time step size, and up to 2000x compared to FOMs, depending on the number of modes used and the flow configuration, without a penalty in accuracy.

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Future work

- Extend the method to hyperreduced ROMs.
- Extend the method to other types of ROMs and flows.

Thank you!

The contents of this presentation are extended in the following paper:

J. Plana-Riu, H. Rosenberger, B. Sanderse, and F.X. Trias, "Stable self-adaptive timestepping for reduced order models for incompressible flows", *Journal of Computational Physics*, 563, 2026, 115164.



Stable self-adaptive timestepping for reduced order models for incompressible flows

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Reduced-order modeling
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Self-adaptive time integration

ABSTRACT

This work introduces RedEigCD, the first self-adaptive timestepping technique specifically tailored for reduced-order models (ROMs) of the incompressible Navier-Stokes equations that employ structure-preserving discretizations. Building upon linear stability concepts, the method adapts the timestep by directly bounding the stability function of the employed time integration scheme using exact spectral information of matrices related to the reduced operators. Unlike traditional error-based adaptive methods, RedEigCD relies on the eigenbounds of the convective and diffusive ROM operators, whose computation is feasible at reduced scale and fully preserves the online efficiency of the ROM. A central theoretical contribution of this work is the proof, based on the combined theorems of Bendixson and Rao, that, under linearized assumptions, the maximum stable timestep for projection-based ROMs is shown to be larger than or equal to that of their corresponding full-order models (FOMs).