

ASSESSING THE IMPACT OF BULK VISCOSITY IN COMPRESSIBLE TURBULENCE

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Summary. *Over the last 70 years, Computational Fluid Dynamics researchers have routinely invoked Stokes' hypothesis to set the bulk viscosity equal to zero. In this work, we aim to analyze the effect of including the second viscosity in the Compressible Navier-Stokes equations by performing a DNS of the Decay and Forcing of homogeneous isotropic turbulence. Initial results indicate the existence of both a non-negligible diffusion effect and the existence of a maximum dissipation regime at a given viscosity ratio.*

1 INTRODUCTION

The viscous stresses are linked with the rate of strain via a constitutive model. For a Newtonian fluid, as the majority of the formulations used in the literature, this link is expressed by the equation:

$$\tau = \mu \left(\nabla u + (\nabla u)^T - \frac{2}{3}(\nabla \cdot u)I \right) + \mu_B(\nabla \cdot u)I, \quad (1)$$

where, following the usual notation, ∇ is the gradient operator, $\nabla \cdot$ the divergence operator, u is the velocity vector, μ and μ_B indicate the dynamic viscosity and the second, bulk or volumetric viscosity, respectively, and I is the identity matrix.

Over the last 70 years, Computational Fluid Dynamics researchers have routinely invoked Stokes' hypothesis [1] to set the volumetric viscosity to zero, in order to omit the last term of the equation, i.e. $\mu_B(\nabla \cdot u)I = 0$. For incompressible fluids or monoatomic gases, this is true. Whether the volumetric viscosity is truly zero, in monoatomic gases, or the velocity divergence is null, in incompressible cases, the Stokes' hypothesis is exact. However, for compressible polyatomic gases, in which the velocity divergence can be a non-negligible value, the second viscosity can be several orders of magnitude higher than the dynamic viscosity.

To give some examples, O_2 or N_2 , the main components of air and polyatomic gases, have volumetric viscosity values of the same order of magnitude as the dynamic viscosity. However, for CO_2 , the volumetric viscosity is approximately 3 orders of magnitude higher than the dynamic one. In other words, three of the most common gases in our atmosphere have a non-zero volumetric viscosity.

Previous work by Pan and Johnsen [2] concluded that the inclusion of bulk viscosity increases the decay rate of turbulent kinetic energy. Nonetheless, they also pointed out that, for air, whose second-to-kinematic viscosity ratio is close to one, and gases with a low bulk-to-kinematic viscosity ratio, neglecting the contribution of the volumetric viscosity would have negligible effects on the numerical results. On the other hand, in gases with a high bulk-to-kinematic viscosity ratio, such as CO_2 , assuming Stokes' hypothesis could lead to inaccurate results.

Nonetheless, all the previously extracted conclusions are based on only a few bulk-to-kinematic viscosity ratios and only in the case of Decaying Homogeneous Isotropic Turbulence. For example, it was concluded that increasing the bulk-to-kinematic viscosity ratio dampened the dilatational component of the turbulent kinetic energy; therefore, the compressible fluid behaved like an incompressible one. But what would happen if energy were to be constantly added? And is this always true, or is there some interval in which the dilatational component increases?

In this work, we analyze the effect of including the second viscosity in the Compressible Navier-Stokes equations. The main objective is to assess how the inclusion of this additional term in the viscous stress tensor affects the flow dynamics and the relation between the different dissipation mechanisms. Furthermore, we want to quantify whether there is a net increase in dissipation, providing a more robust and physically meaningful option for ensuring numerical stability in compressible simulations than the usual upwind-based approaches.

To achieve our goal, we have divided the present work into two main blocks. In the first block, the usual Decay of Homogeneous Isotropic Turbulence will be analyzed, with a broader range of bulk-to-kinematic viscosity ratios included in the discussion. In the second part, we will analyze whether the conclusions drawn from a Decaying case can be safely extrapolated into a Forced one. Regarding the second part of the work, the authors believe that analyzing what occurs in a problem in which energy is continuously injected, akin to a typical real-world scenario where energy is injected via boundary conditions, could shed some light on the energy-transfer phenomena involving the usually omitted volumetric viscosity.

2 NUMERICAL METHOD

When solving the Compressible Navier-Stokes equations numerically, it is possible to obtain solutions without any physical sense. Density, temperature, and pressure can take only positive values to have physical meaning. Nonetheless, in numerical simulations, they can be assigned negative values. When the simulation reaches negative values, it usually crashes, as imaginary numbers start to appear. To overcome this problem, we use logarithmic variables as in Shebalin [3]: $\rho = e^\lambda$ and $T = e^\sigma$. With this change, ρ and T are replaced by variables λ and σ , respectively.

These new variables can have negative values during the simulation. However, the primitive variables, ρ and T , can no longer be negative as $e^x > 0; \forall x \in \mathbb{R}$. This change modifies the usual formulation of the Navier-Stokes equations to:

$$\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = -\nabla \cdot u \quad (2)$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\gamma} e^\sigma \nabla (\lambda + \sigma) + e^{-\lambda} \nabla \cdot \tau \quad (3)$$

$$\frac{\partial \sigma}{\partial t} + u \cdot \nabla \sigma = -(\gamma - 1) \nabla \cdot u + \kappa e^{-\lambda} (\nabla^2 \sigma + (\nabla \sigma)^2) + \gamma(\gamma - 1) e^{-\lambda - \sigma} \phi \quad (4)$$

where all the variables, i.e. λ , u , and σ , are non-dimensional, γ is the heat capacity ratio, κ is the non-dimensional heat conductivity, and ϕ is the viscous dissipation rate.

Since the previous equations cannot be analytically solved for a general case but numerically, we need to select a numerical method. Finite Element, Difference, or Volume Methods resort to numerically approximating the derivatives that appear in the set of equations. Nonetheless, this approximation leads to dispersion and, if not symmetric schemes are used, to diffusion errors. To keep those errors up to a minimum, and given that the selected problem has periodic boundary conditions on all its boundaries, the pseudospectral method can be used. Therefore, all the derivatives are evaluated in the Fourier space as:

$$\hat{\phi}'(k) = k \hat{\phi}(k), \quad (5)$$

. Then, they are transformed into the physical space via the inverse Fourier transform. Once in the physical space, the products between the derivatives and the rest of the variables, such as $u \cdot \nabla \sigma$, are evaluated.

3 CASE SETUP

As commented in the introduction, in this work, we will study both the Decayed and Forced Homogeneous Isotropic Turbulence cases. We define a box-sized domain discretized in $M_x \times M_y \times M_z$ grid points in physical space, and $N_x \times N_y \times N_z$ in Fourier space. We have used a 3/2 rule to avoid aliasing between the number of nodes and modes at a given direction, i.e., $M_i = 3/2 N_i$. For all the present simulations, the number of modes at each direction is fixed at $N_x = N_y = N_z = 128$.

An initial Re_λ of 100 is achieved by fixing the kinematic viscosity. The bulk viscosity is fixed via the ratio with the kinematic viscosity, i.e., μ_B/μ . An initial turbulent Mach number, M_t , is fixed by setting the initial temperature field such that the relation: $M_t = \sqrt{\frac{\langle u^2 \rangle}{\langle T \rangle}}$ is fulfilled. Finally, the Pr number is held constant at 0.7 across all configurations studied. Finally, for the Forced test cases, energy is injected into the system by fixing the amplitude of the first two Fourier modes.

All the Fast Fourier Transform tools used within this work, as well as all the functions required to post-process the results obtained, are available in an online repository [4]. The purpose of the present work is also to extend this repository's current capabilities to compressible flows.

4 DECAYED HOMOGENEOUS ISOTROPIC TURBULENCE

First, we will analyze the obtained results for the Decayed Homogeneous Isotropic Turbulence case. The first parameter analyzed is how the initial Re_λ and M_t numbers decay over time. In Figure 1, we can see how, as expected, the initial Reynolds number based on the Taylor microscale decays over time. Nonetheless, two different results can be interpreted from that figure. The first is that, for increased bulk-to-kinematic viscosity ratios, the final values at the end of the simulation are greatly reduced, and the second is that, when increasing the initial turbulent Mach numbers, fluctuations in their magnitude are greatly attenuated. Similar conclusions can be extracted from Figure 2.

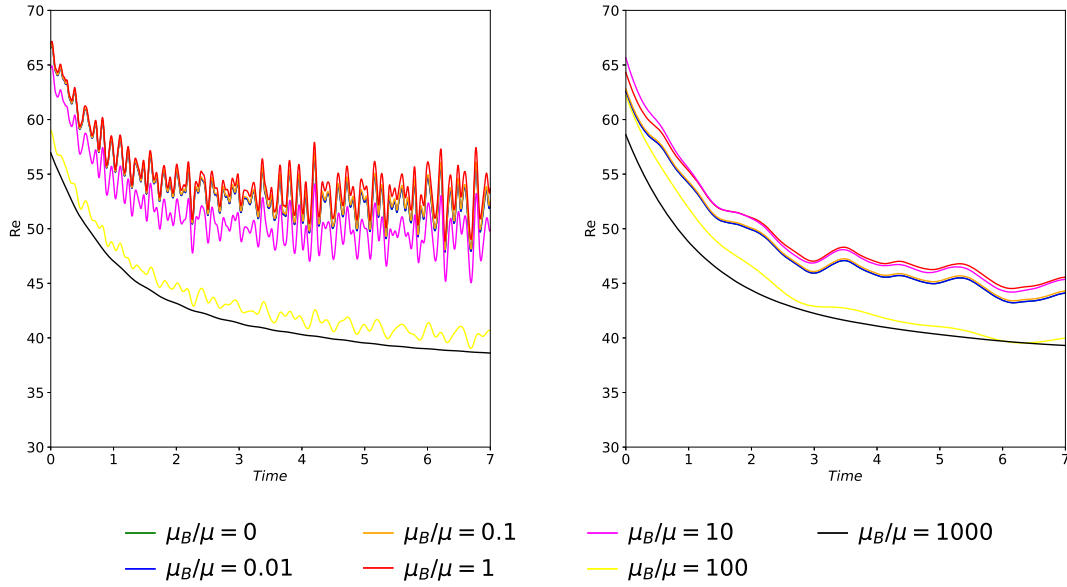


Figure 1: Time evolution of Re_λ . Left: Initial $M_t = 0.1$; Right: Initial $M_t = 0.8$

This is not surprising, as an increased bulk-to-kinematic viscosity ratio effectively adds more diffusion to the numerical simulation. This increase results in a faster decay of the kinetic energy and, consequently, in its related magnitudes, i.e., Re_λ and M_t . Related to the magnitude fluctuations, these are due to the temporal mismatch between the hydrodynamic and acoustic scales; when this mismatch is reduced (i.e., high initial turbulent Mach number), both scales have the same order of magnitude.

The second analysis done in this section is devoted to the different dissipation mechanisms present in the simulation. To achieve this goal, we put our focus on two different ratios: the bulk-to-shear dissipation ratio, $(\mu_B/\mu)\langle\nabla\cdot u\rangle^2/(2S_{ij}^d S_{ij}^d)$, and the dilatational-to-solenoidal dissipation ratio, $(4/3 + \mu_B/\mu)\langle\nabla\cdot u\rangle^2/\langle\omega\omega\rangle$. This methodology allows us to understand how the different dissipation mechanisms are affected by the presence of bulk viscosity.

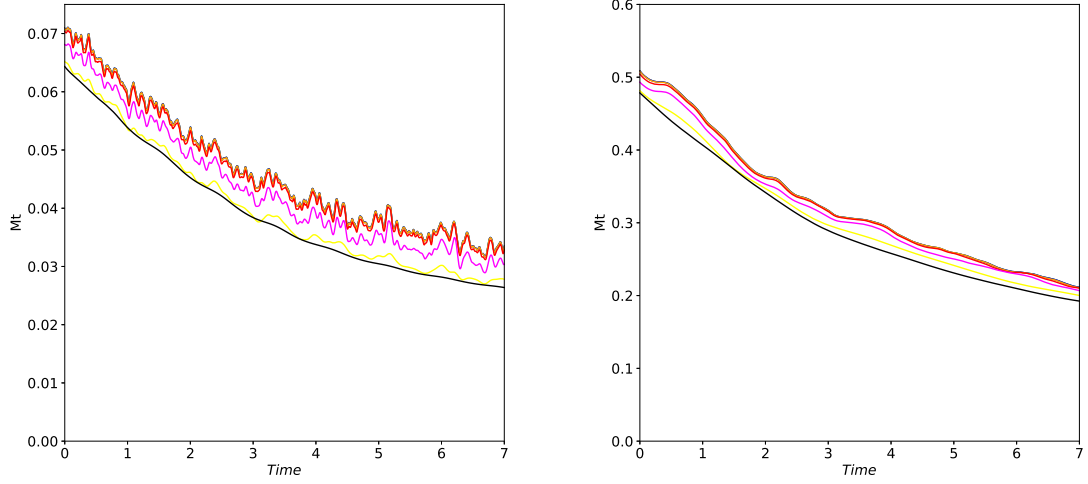


Figure 2: Time evolution of turbulent Mach number, M_t .
Left: Initial $M_t = 0.1$; Right: Initial $M_t = 0.8$

The first thing that can be noticed in Figure 3 is that there is a "maximum rate" at the end of the simulation depending on the bulk-to-kinematic viscosity ratio. This ratio starts at zero when μ_B/μ is null due to Stokes' hypothesis, but even when it is maximized, i.e., 1000, it does not reach the maximum value.

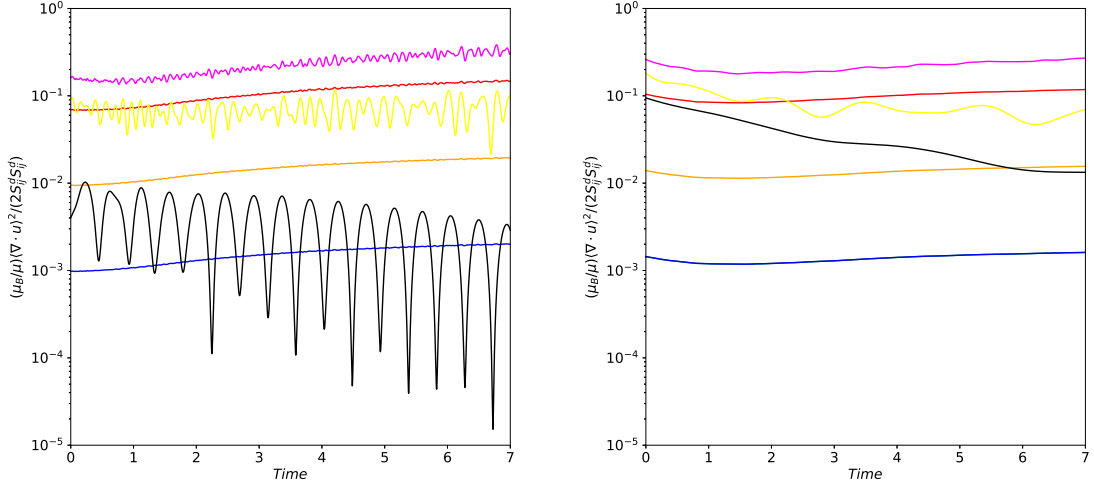


Figure 3: Time evolution of bulk-to-shear dissipation ratio.
Left: Initial $M_t = 0.1$; Right: Initial $M_t = 0.8$

Table 1: Bulk-to-shear dissipation ratio, $\frac{(\mu_B/\mu)\langle\nabla\cdot u\rangle^2}{(2S_{ij}^d S_{ij}^d)}$, at the end of the simulation for different initial turbulent Mach numbers, M_t

M_t	μ_B/μ	0	0.01	0.1	1	10	100	1000
0.1		0	0.002	0.019	0.147	0.303	0.1	0.003
0.5		0	0.002	0.017	0.127	0.290	0.063	0.005
0.8		0	0.002	0.016	0.118	0.270	0.07	0.013

On the other hand, in the case of the dilatational-to-solenoidal dissipation ratio and due to the grouping of results at the end of the simulation, as shown in Figure 4, it is more difficult to conclude that there is a maximum dissipation ratio at a given μ_B/μ . Nonetheless, zooming in reveals that there is, in fact, another maximum when Stokes' hypothesis is omitted.

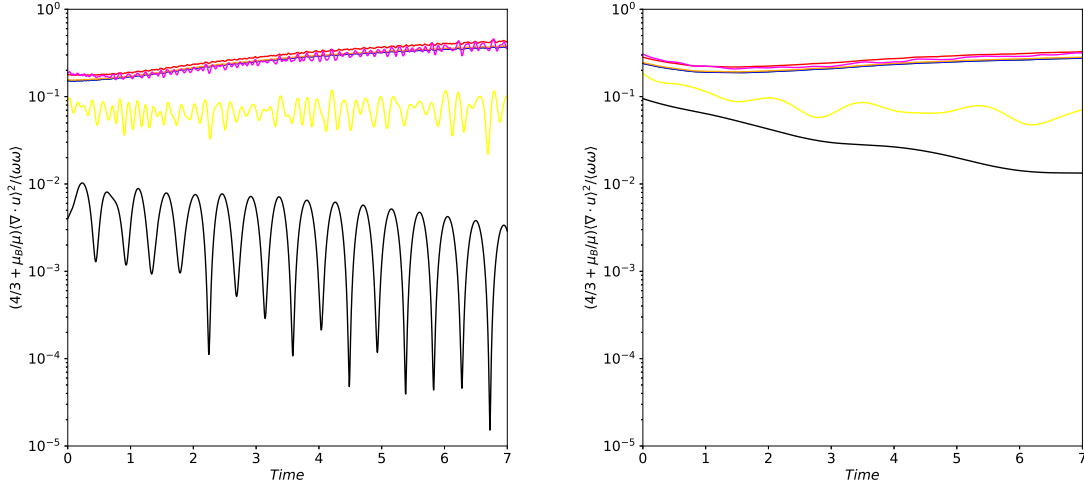


Figure 4: Time evolution of dilatational-to-solenoidal dissipation ratio.

Left: Initial $M_t = 0.1$; Right: Initial $M_t = 0.8$

To make the results more understandable, we have analyzed the previously presented ratios at the end of the simulation and included them in Tables 1 and 2. As can be seen, both ratios, independent of the initial turbulent Mach number, have a maximum around μ_B/μ equal to 1 and 10. Therefore, it seems that around those viscosity ratios, both the bulk-to-shear dissipation ratio and the dilatational-to-solenoidal dissipation ratio are maximized.

As the final analysis in this section and to understand why these maxima appear, we will study how the different elements that define these dissipation ratios, i.e., the dilatation and the enstrophy, evolve in time.

Table 2: Dilatational-to-solenoidal dissipation ratio, $\frac{(4/3 + \mu_B/\mu)\langle \nabla \cdot u \rangle^2}{\langle \omega \omega \rangle}$, at the end of the simulation for different initial turbulent Mach numbers, M_t

M_t	μ_B/μ	0	0.01	0.1	1	10	100	1000
0.1		0.367	0.368	0.377	0.428	0.356	0.101	0.003
0.5		0.301	0.302	0.309	0.357	0.342	0.064	0.005
0.8		0.275	0.276	0.282	0.327	0.317	0.071	0.013

In Figure 5, we have included how the divergence of the velocity field evolves in time. The consequence of including the second viscosity term in the Navier-Stokes equations is immediately extracted from this figure: the dilatation is greatly reduced. As can be seen, the velocity divergence is reduced by several orders of magnitude in the highest viscosity ratio considered here. Therefore, even though we are dealing with compressible fluids, they start to behave like incompressible ones.

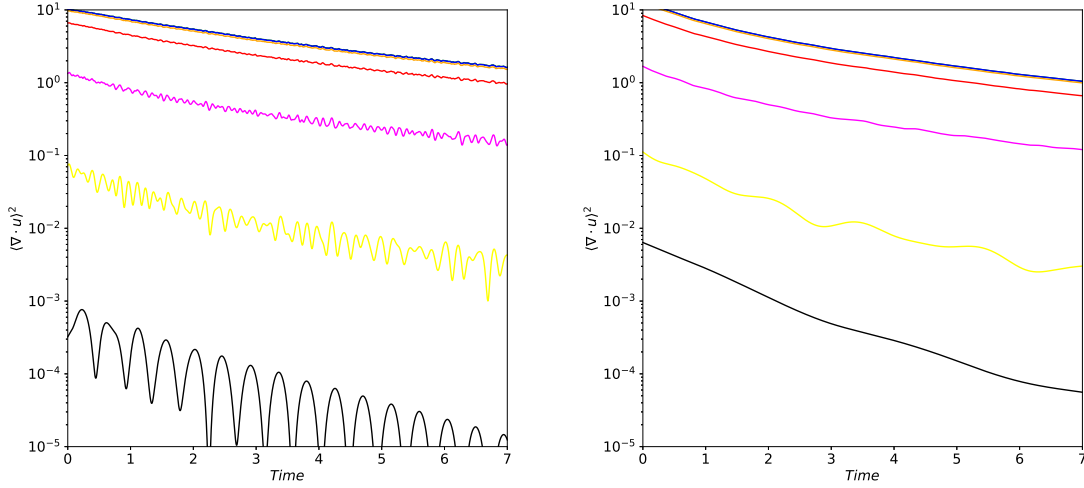


Figure 5: Time evolution of dilatation.
Left: Initial $M_t = 0.1$; Right: Initial $M_t = 0.8$

In Figure 6, we have included the temporal evolution of the enstrophy at different initial turbulent Mach numbers. In this case, unlike the dilatation field, the consequence is not obvious. At first glance, it may seem that including the second viscosity has a negligible effect, as all the profiles seem to converge towards the same plot. To make the results more understandable, and as we did before, we have included dilatation and enstrophy at the end of the simulations in Tables 3 and 4.

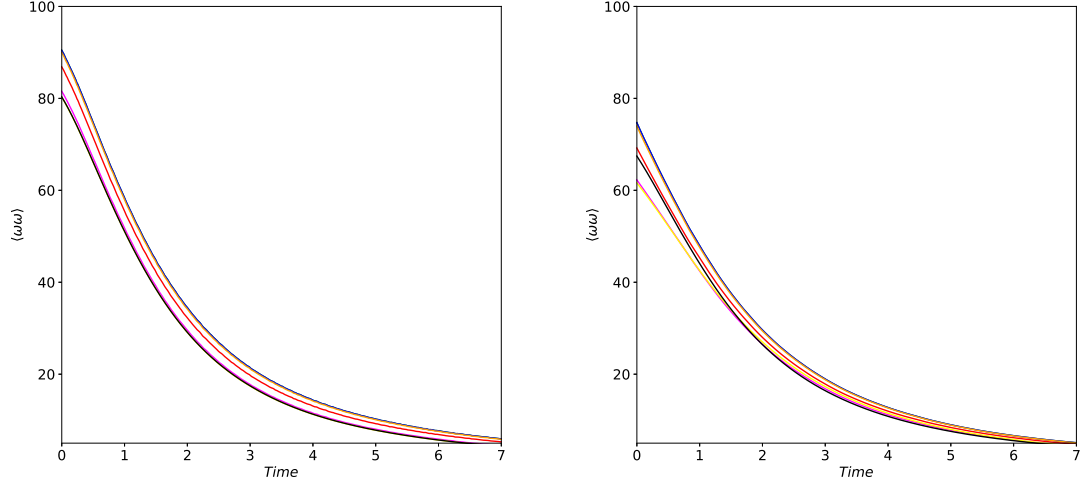


Figure 6: Time evolution of enstrophy. Left: Initial $M_t = 0.1$; Right: Initial $M_t = 0.8$

Table 3: Dilatation at the end of the simulation

M_t	μ_B/μ	0	0.01	0.1	1	10	100	1000
0.1		1.633	1.622	1.533	0.965	0.140	0.004	1e-5
0.5		1.185	1.180	1.120	0.737	0.130	0.003	2e-5
0.8		1.05	1.05	0.995	0.662	0.120	0.003	5e-5

As shown in Table 3, although $\nabla \cdot u$ is effectively becoming smaller as bulk-to-kinematic viscosity ratios increase, their product is still increasing. This product is, indeed, maximized around μ_B/μ equal 10, which matches the maxima previously observed in the bulk-to-shear dissipation ratios. Regarding enstrophy, as shown in Table 4, it is indeed affected by the presence of the second viscosity. Its levels are reduced by up to 30% at lower initial turbulent Mach numbers and higher maximum viscosity ratios. Nonetheless, This reduction is far from the one experienced by the dilatation field previously commented.

5 FORCED HOMOGENEOUS ISOTROPIC TURBULENCE

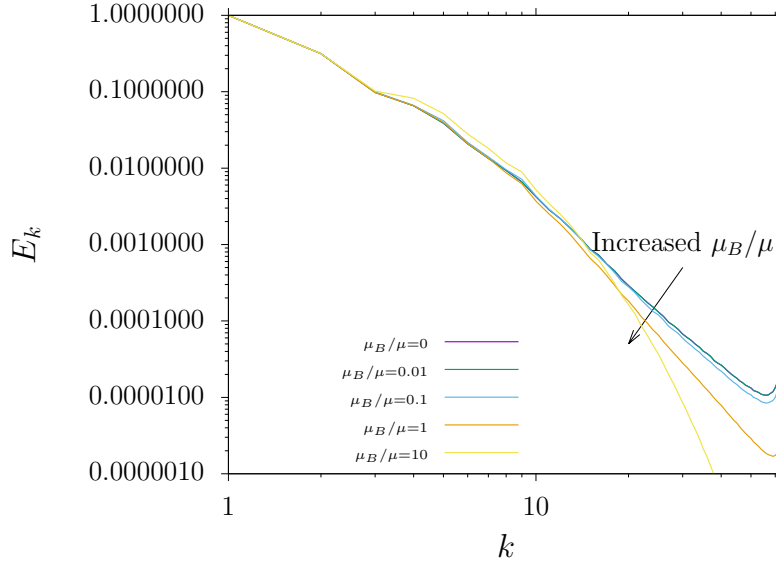
Results in the previous section have shown the existence of a bulk-to-kinematic viscosity ratio that maximizes both the dilatational-to-solenoidal dissipation ratio and the bulk-to-shear dissipation ratio. Nonetheless, the values at the end of the previous simulations exhibit temporal oscillations that may have led to incorrect conclusions. In other words, the differences between results across different bulk-to-kinematic viscosity ratios can be attributed to temporal oscillations rather than to differences between averaged values.

Table 4: Enstrophy at the end of the simulation

M_t	μ_B/μ	0	0.01	0.1	1	10	100	1000
0.1		5.934	5.924	5.834	5.265	4.432	4.280	4.282
0.5		5.252	5.246	5.188	4.822	4.290	4.326	4.284
0.8		5.10	5.095	5.047	4.727	4.303	4.337	4.200

To make this comparison fairer, the Forced Homogeneous Isotropic Turbulence test case is considered, since the values reported at the end of the simulation are no longer instantaneous fields, as in the previous section, but temporally averaged ones. For the sake of brevity, only results at an initial turbulent Mach number of 0.1 are presented.

The kinetic energy cascade at different bulk-to-kinematic viscosity ratios is shown in Figure 7. The effect of adding the bulk viscosity to the Navier-Stokes equations is only visible at high viscosity ratios and higher frequencies, as the tail of the kinetic energy cascade differs across the different cases considered. The inertial subrange appears to be very similar across all the simulations performed.

**Figure 7:** Kinetic energy cascade for the Forced Homogeneous Isotropic Turbulence

The temporally averaged parameters at the end of the simulations are summarized in Table 5. As can be seen, both the initial Re_λ and M_t are reduced from the imposed values. This is explained by the initial velocity and temperature fields developing, effectively modifying both Re_λ and M_t . Dilatation is reduced by an order of magnitude between the case assuming the Stokes' hypotheses and the case with the highest viscosity ratio considered here.

This perfectly matches the conclusions drawn in the previous section, together with the fact that, even though dilatation is reduced, its product with the bulk-to-kinematic viscosities is increased. Enstrophy, on the other hand, shows a lesser variation in comparison with the Decayed case, showing a decrease lesser than 10% in comparison with previously values around 20-30%.

Table 5: Average values for the Forced Compressible Homogeneous Isotropic Turbulence case at different μ/μ_B ratios

μ_B/μ	0	0.01	0.1	1	10
Re_λ	92.93	91.57	92.37	94.63	89.09
M_t	0.09	0.091	0.092	0.09	0.085
$\langle(\nabla \cdot u)^2\rangle$	11.796	11.154	10.82	9.07	2.10
$\langle\omega\omega\rangle$	22.435	22.231	21.982	21.596	20.509
$\langle p\nabla \cdot u\rangle$	-0.0029	0.093	0.044	-0.064	0.019
$(\mu_B/\mu)\langle(\nabla \cdot u)^2\rangle/(2S_{ij}^d S_{ij}^d)$	0	0.003	0.031	0.267	0.718
$(4/3 + \mu_B/\mu)\langle(\nabla \cdot u)^2\rangle/\langle\omega\omega\rangle$	0.649	0.716	0.823	0.972	0.902

Finally, regarding the dissipation ratios presented, it can be seen that previous extracted conclusions hold. Therefore, in the case of the Decayed Homogeneous Isotropic Turbulence, the maxima found are not attributed to the temporal oscillations observed. Again, the bulk-to-viscous dissipation ratio shows a maximum at a bulk-to-kinematic viscosity ratio of 10. And the dilational-to-solenoidal dissipation ratio shows a maximum around a bulk-to-kinematic viscosity ratio of 1. Therefore, and as a first initial guess, it may seem that between μ_B/μ 1 and 10 there is a viscosity ratio that could maximize both dissipation ratios.

6 CONCLUSIONS

The numerical results of both the Decayed and the Forced Homogeneous Isotropic Turbulence at different turbulent Mach numbers and bulk-to-kinematic viscosity ratios have been presented.

Regarding the Decayed case, almost all the studied magnitudes in this work have shown dependence with the bulk viscosity. Except for enstrophy, which has remained almost constant with the inclusion of the bulk viscosity in the Navier-Stokes equations, the rest of the parameters have shown results that depend on the presence of the bulk viscosity. Each of the studied parameters has also presented dependence on the initial turbulent Mach number, showing fluctuations that are substantially reduced when this parameter increases. In the highest-considered bulk-to-kinematic-viscosity case, the Reynolds number based on the Taylor microscale is 10% smaller at the end of the simulation than in the case assuming Stokes' hypotheses. The divergence of the velocity field strongly depends on the value of the bulk viscosity, as it has been shown that the fluid starts to behave like an incompressible fluid, even though a fully compressible approach has been taken.

Both the dissipation ratios considered in this work, the bulk-to-shear dissipation, $(\mu_B/\mu)\langle\nabla\cdot u\rangle^2/(2S_{ij}^d S_{ij}^d)$, and the dilatational-to-solenoidal dissipation ratio, $(4/3 + \mu_B/\mu)\langle\nabla\cdot u\rangle^2/\langle\omega\omega\rangle$, have shown a maximum value at the end of the simulation which is not achieved at 0 bulk viscosity, i.e., Stokes hypothesis, nor at maximum value. Therefore, this may suggest the existence of a maximum dissipation regime located between μ_B/μ 1 and 10.

In order to ensure that the maximum values found in the Decayed case are not due to instantaneous fluctuation, achieving instantaneous higher values, but not higher averaged ones, the Forced case was considered. Nonetheless, all the conclusions extracted in the first part are consistent with the ones extracted on this part. Enstrophy is mainly unaffected by the presence of the bulk viscosity, showing only a difference of less than 10% between the highest bulk viscosity test case and the one assuming the Stokes hypothesis. Dilatation, on the other hand, still exhibits a strong dependence on the added bulk viscosity, showing a considerable reduction when this parameter is added. Finally, the previously detected interval between μ_B/μ 1 and 10 is still found in this case. In this interval, both the considered dissipation ratios reach the maximum values.

The inclusion of the bulk viscosity in the Navier-Stokes equations has been shown to affect the flow topology of the resulting solution. This effect has, a priori, only positive aspects. Firstly, it reduces the dilatation field of the obtained flow, making the fluid to behave like a more incompressible one. Second, it increases the amount of diffusion added by about 15-20%. Nonetheless, this is a completely free, physically based increase in dissipation. Therefore, it may open the possibility of reducing reliance on upwind-based approaches within the compressible community and using this approach instead as a more physically sensible alternative.

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