On the Evaluation of Spatial and Temporal Numerical Dissipation in Incompressible Flows

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3. Methodology
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5. Conclusions and future work

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Navier-Stokes Equations

Continuous Formulation

Incompressible flows with constant physical properties:

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \mu \nabla^2 \boldsymbol{u} \ + \boldsymbol{S}$$

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Matrix-Vector Formulation

Spatially discretized algebraic system of equations:

$$\mathbf{M}\boldsymbol{u_s} = \mathbf{0_c}$$

$$\Omega \frac{d\boldsymbol{u_c}}{dt} + \mathbf{C}(\boldsymbol{u_s})\boldsymbol{u_c} = \mathbf{D}\mathbf{u_c} - \Omega \mathbf{G_c}\boldsymbol{p_c} + \mathbf{f_c}$$

Kinetic energy equation

- The total kinetic energy of the system is defined as:

Continuous Formulation

$$E_k = \int_\Omega \frac{1}{2} u^2 dV$$

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Matrix-Vector Formulation

$$E_k = \frac{1}{2}\langle \boldsymbol{u}_c, \boldsymbol{u}_c \rangle = \frac{1}{2}\|\boldsymbol{u}_c\|^2 \in \mathbb{R}$$

where $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \boldsymbol{a}^T \Omega \boldsymbol{b}$ is the discrete inner-product of vectors. Therefore,

$$E_k = \frac{1}{2}\boldsymbol{u}_c^T \Omega \boldsymbol{u}_c$$

Kinetic energy equation

- Then, the time evolution of E_k is obtained by taking the inner-product of the discrete velocity field, $\boldsymbol{u}_c$, and the momentum equation given in the momentum NS equation.

$$\frac{dE_k}{dt} = -\boldsymbol{u}_c^T \mathbf{C}(\boldsymbol{u}_s)\boldsymbol{u}_c + \boldsymbol{u}_c^T \mathbf{D}\boldsymbol{u}_c - \boldsymbol{u}_c^T \Omega \mathbf{G}_c \boldsymbol{p}_c + \boldsymbol{u}_c^T \Omega \boldsymbol{f}_c$$

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Energy conserving formulation

- Looking back at the momentum NS equation in matrix-vector form:

$$\Omega \frac{d\boldsymbol{u_c}}{dt} + \mathbf{C}(\boldsymbol{u_s})\boldsymbol{u_c} = \mathbf{D}\mathbf{u_c} - \Omega \mathbf{G_c}\boldsymbol{p_c} + \mathbf{f_c}$$

- we can see that in the absence of external sources, the rate of change of total energy is neither influenced by convective transport nor by pressure differences; it is solely determined by dissipation:

$$\frac{\mathrm{d}E_k}{\mathrm{d}t} = \left\langle \boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial t} \right\rangle = -\nu \|\nabla \boldsymbol{u}\|^2$$

Energy conserving formulation

Spatial Operator Properties

- **Incompressibility ($\nabla \cdot \boldsymbol{u} = 0$):** Pressure term vanishes via integration by parts.
- **Skew-symmetry:** Convection operator $C(\boldsymbol{u}, \cdot)$ conserves kinetic energy:

$$\langle C(\boldsymbol{u}, \phi), \phi \rangle = 0$$

Skew-symmetry of the convective term

- Finally, the skew-symmetry of $C(\boldsymbol{u}_s)$ and the duality between gradient and divergence operator leads to null contributions of the convective and pressure gradient term.
- Therefore, in the absence of external forces or contributions from the domain boundary, we can obtain:

$$\frac{dE_k}{dt} = \boldsymbol{u}_c^T D \boldsymbol{u}_c \leq 0$$

Energy conservation

To ensure that the discrete Navier–Stokes equations respect the same physical conservation laws as the continuous system, the discrete operators must preserve the structural properties of the corresponding differential operators:

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- **Convective operator:** Skew-symmetric
- **Diffusive operator:** Symmetric negative definite
- **Gradient and divergence operators:** Negative adjoints

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- How can we identify and quantify the non-physical dissipation of energy?
- How do we know if a simulation's accuracy is being ruined by spatial discretization vs. temporal time-stepping errors?
- What practical framework can guide us in selecting and validating time-integration schemes for non-linear flows?

Energy conservation

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- To have a way to neatly evaluate the effect of the SGS models and how can the numerical errors interfere.

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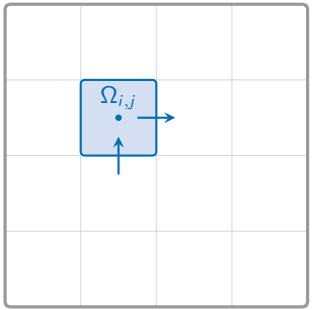
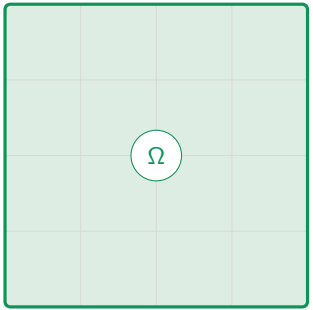
Global vs. Local energy budgets



Local EB approach



Global EB approach

Global vs. Local energy budgets

- Calculation of the dissipative term for the constant viscosity field

Global approach

$$\left.\frac{dE_k}{dt}\right|_{diff} = \boldsymbol{u}_c^T D \boldsymbol{u}_c$$

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Local approach

$$\left.\frac{dE_k}{dt}\right|_{diff} = -\nu \boldsymbol{\Pi}_{s\to c} \Omega_s \mathbf{G_s u_c} : \mathbf{G_s u_c}$$

where $\boldsymbol{\Pi}_{s\to c}$ is a face-to-node interpolator

Evaluating temporal errors

Current time step

General formula at the current time step t^{n+1}

$$\left.\frac{\mathrm{d}E_k}{\mathrm{d}t}\right|^{n+1} = \left(-\boldsymbol{u}_c^T\mathbf{C}(\boldsymbol{u}_s)\boldsymbol{u}_c + \boldsymbol{u}_c^T\mathbf{D}\boldsymbol{u}_c - \boldsymbol{u}_c^T\Omega\mathbf{G}_c\boldsymbol{p}_c + \mathbf{u_c}^T\Omega\mathbf{f_c}\right)^{n+1}$$

Evaluating temporal errors

Time-integration scheme adapted - Multi-step methods

$$\frac{E_k^{n+1/2+\kappa} - E_k^{n-1/2+\kappa}}{\Delta t} = (\mathbf{u_c}^{n+\kappa})^T \Omega \mathbf{R} \tilde{\mathbf{u}}_c^{n+\kappa}$$

where

$$\tilde{\mathbf{u}}_c^{n+\kappa} \equiv (1+\kappa)\mathbf{u}_c^n - \kappa \mathbf{u}_c^{n-1}$$

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Time-integration scheme adapted - Multi-stage methods

$$\frac{E_k^{n+1} - E_k^n}{\Delta t_n} = \left(\frac{\boldsymbol{u}_c^{n+1} + \boldsymbol{u}_c^n}{2} \right)^T \Omega \left(\sum_{i=1}^{s} b_i R(\boldsymbol{U}_i) \right)$$

Preserving energy

- In buoyancy-driven flows, we have:

$$\frac{dE_{kinetic}}{dt} + \frac{dE_{potential}}{dt} + \frac{dE_{internal}}{dt} = 0$$

$$Potential\ energy \Rightarrow Kinetic\ energy \Rightarrow Thermal\ (internal)\ energy$$

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Results: Rayleigh–Taylor Flow

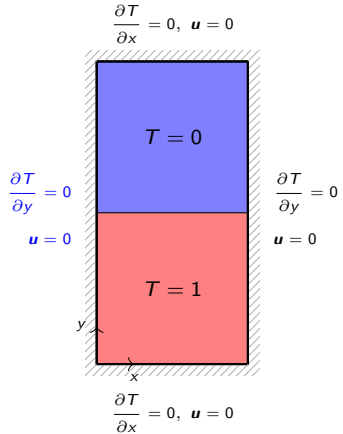

Figure 1. The Rayleigh–Taylor case setup

Results: Rayleigh–Taylor Flow

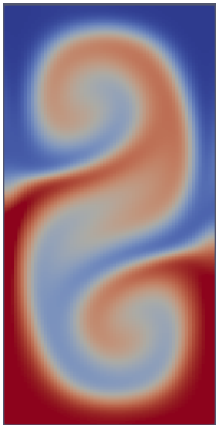

Figure 2. Instantaneous temperature fields in Rayleigh–Taylor case

Results: Rayleigh–Taylor Flow

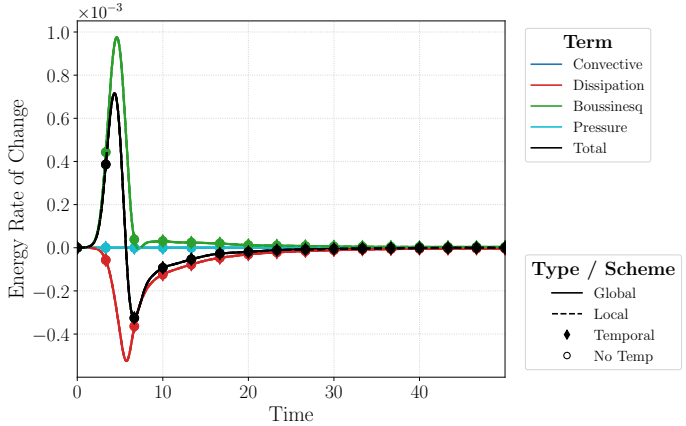

Figure 3. Energy budgets for Rayleigh–Taylor flow at $Ra = 10^6$ and $Ge = 1$ ($A = 2,\ 128 \times 256$)

Results: Rayleigh–Taylor Flow

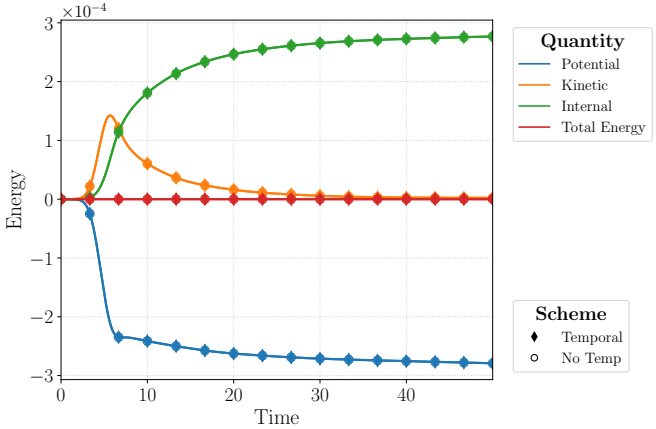

Figure 4. Time evolution of kinetic, potential, and internal energy budgets ($Ra = 10^6,\ Ge = 1, A = 2,\ 128 \times 256$)

Results: Rayleigh–Taylor Flow

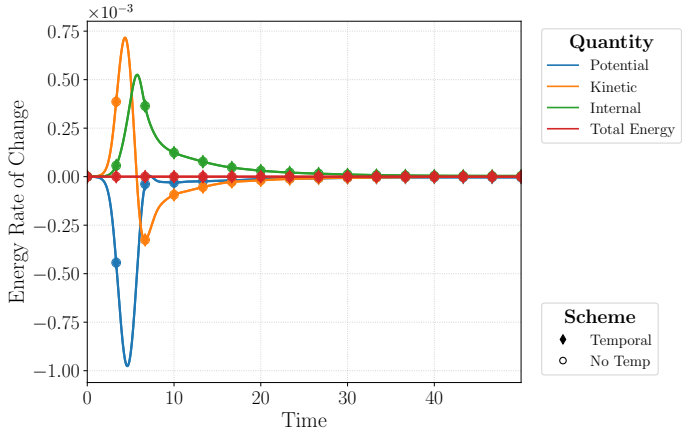

Figure 5. Rate of change of kinetic, potential, and internal energies ($Ra = 10^6,\ Ge = 1, A = 2,\ 128 \times 256$)

Results: Channel Flow

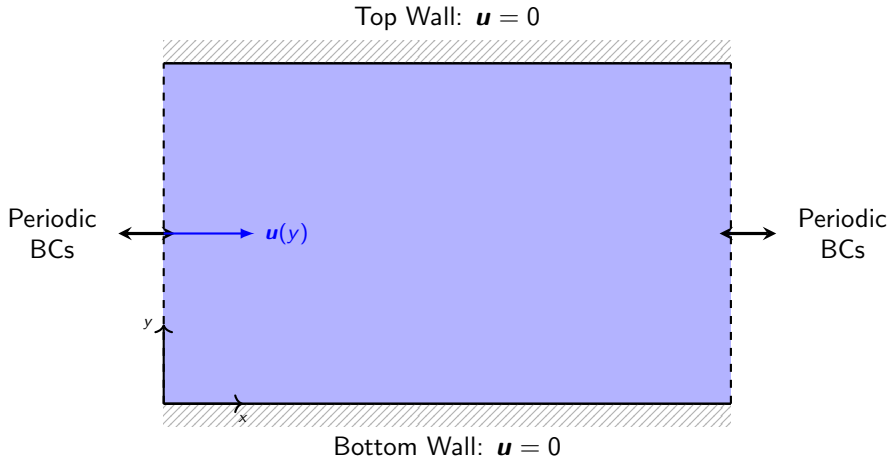

Figure 6. The Channel Flow case setup

Results: Channel Flow

- Calculation of numerical dissipation in LES simulation:

$$\varepsilon_{\text{total}} = \varepsilon_{\text{physical}} + \varepsilon_{\text{SGS model}} + \varepsilon_{\text{numerical dissipation}}$$

Results: Channel Flow

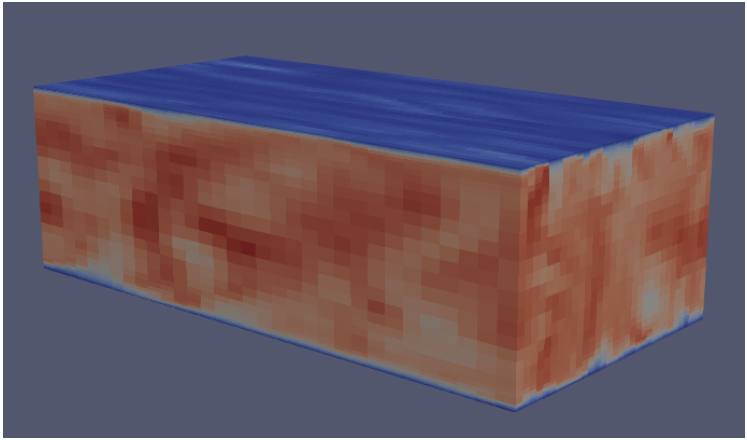

Figure 7. Instantaneous streamwise velocity field in Channel Flow

Results: Channel Flow

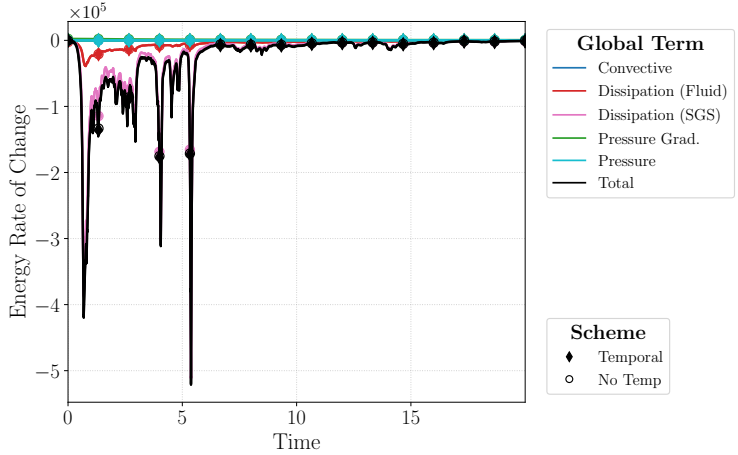

Figure 8. Energy Budgets in Channel Flow case using LES model ($Re_\tau = 180,\ 32 \times 32 \times 32$)

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Conclusions

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- **Spatial vs. Temporal Error Disentanglement:** Energy budgets successfully isolate spatial discretization errors from time-integration scheme dissipation.
- **Local vs. Global Insights:** Local energy budgets reveal regional numerical anomalies (e.g., near boundaries or high gradients) missed by global metrics.

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- **Structural Conservation:** Preserving discrete operator symmetry (skew-symmetry, negative adjointness) is crucial to eliminate non-physical energy sources.
- **Spatial vs. Temporal Error Disentanglement:** Energy budgets successfully isolate spatial discretization errors from time-integration scheme dissipation.
- **Local vs. Global Insights:** Local energy budgets reveal regional numerical anomalies (e.g., near boundaries or high gradients) missed by global metrics.
- **Model Validation Tool:** Provides a rigorous, parameter-free framework for validating time-step selection and SGS models in complex flows.

Future work

- Creating a universal approach that can be applicable to various cases

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- Coupling with additional physical phenomena

Thank you for your attention!
Questions?