

ADAPTIVE MOVING MESH FOR CONSERVATIVE MULTIPHASE FLOWS DISCRETIZATION

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Summary. *This work presents a conservative discretization framework for multiphase flows coupled with an adaptive, moving deformative mesh. An algebraic energy-preserving conservative level-set method is combined with a variational, hypoelastically stabilized mesh deformation whose control metric is driven by the level-set field, so that the resulting Arbitrary Lagrangian–Eulerian formulation shares the volume metric between the geometric conservation law and the energy-preserving scheme. The coupled framework is assessed on free-stream preservation, temporal accuracy and the canonical cylindrical column benchmark. The results deliver second-order temporal convergence and confirm the energy-preserving behaviour on a static mesh, while exposing the residual discrete geometric conservation law error that currently bounds the admissible mesh deformation.*

1 INTRODUCTION

Direct Numerical Simulation (DNS) of multiphase flows remains a demanding problem, mainly because of the dynamic interaction between interfacial tension and fluid mechanics and the need to accurately resolve thin, deforming interfaces. Whereas DNS of single-phase turbulence has reached a substantial degree of maturity, interface-resolved multiphase flows lag behind due to two specific difficulties: (i) surface tension establishes a dynamic equilibrium in which kinetic and surface energy are continuously exchanged, and (ii) the discontinuity of the physical properties across the interface constrains how interpolations may be performed without breaking physical laws [1]. An overview of the interface-capturing strategies developed to

address these difficulties can be found in [3] and references therein.

The present framework is built upon the Conservative Level-Set (CLS) method of Olsson and Kreiss [4], selected for its conservation properties, its curvature accuracy and the ease with which it handles topological changes. Nevertheless, capturing the interface accurately is a necessary but not a sufficient condition for a physically consistent simulation. In the spirit of the symmetry-preserving discretizations of Verstappen and Veldman [5], later extended to collocated unstructured grids by Trias et al. [6], the discrete operators must preserve the underlying mathematical structure of the continuous problem. Valle et al. [1] identified the structure associated with the capillary term and proposed a curvature discretization that conserves mechanical energy, i.e. kinetic plus surface, up to temporal integration; the analysis was subsequently extended in [2]. Therefore, the conservative behaviour of the scheme is not an incidental property of a particular mesh, but the consequence of a set of discrete symmetries that any further development must respect.

Moreover, the whole framework is formulated algebraically: every operator is a sparse matrix and the algorithm is a concatenation of matrix–vector products, which delivers portable implementations on heterogeneous high-performance computing systems without compromising physical fidelity [7–10]. It must be noted that this is not a mere implementation preference; it dictates which mesh adaptation strategies remain admissible. Coupling multiphase solvers with mesh adaptation is by no means new, and a rich variety of parallel adaptive mesh refinement (AMR) algorithms is available [11, 12]. However, classical h -adaptive AMR creates and destroys control volumes, which modifies the size and the sparsity pattern of every operator and redistributes the load across the parallel ranks. The question addressed here is consequently not how to refine a mesh, but which adaptation strategy blends cleanly with an algebraic, portable, energy-preserving solver. An r -adaptive (node-relocation) strategy answers that question, since it keeps the topology fixed; among the available options [14], the variational quasi-isometric mesh generator with hypoelastic stabilization of Garanzha and Kudryavtseva [13] was adopted.

Furthermore, moving the mesh invalidates the frame in which the conservation proofs were established. The energy-preserving discretization of [1] assumes a fixed grid, and the Arbitrary Lagrangian–Eulerian (ALE) literature has long shown that a moving-grid scheme only inherits the nonlinear stability of its fixed-grid counterpart when the Discrete Geometric Conservation Law (DGCL) is satisfied [15–18]. Two research questions therefore frame this work: does the coupled scheme remain a conservative framework, and how much does mesh motion interfere with an analysis that is only valid on a static frame?

The contribution of this paper is threefold: (i) an algebraic ALE formulation in which the Geometric Conservation Law (GCL) is expressed through the very same volume metric Ω that supports the energy-preserving discretization, (ii) a Cartesian “accordion” restriction of the variational mesher that reduces the deformation problem to a tridiagonal system while the level-set field drives the control metric, and (iii) a lagged-mesh, second-order Adams–Bashforth integration strategy that reconciles the midpoint requirement of the DGCL with the divergence constraint imposed at the end of the step. The remainder of the paper is arranged as follows. Section 2 presents the continuum model, Section 3 the adaptive moving mesh strategy and Sec-

tion 4 the ALE formulation and its geometric constraint. Results are reported in Section 5 and conclusions are outlined in Section 6.

2 GOVERNING EQUATIONS AND INTERFACE REPRESENTATION

The physical model comprises the incompressible Navier–Stokes equations for a Newtonian fluid,

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = \nabla \cdot \sigma, \quad \nabla \cdot \vec{u} = 0, \quad (1)$$

where ρ is the density, $\vec{u}$ the velocity field and $\sigma = -p\mathbb{I} + \tau$ the stress tensor, split into its hydrostatic and deviatoric parts. The deviatoric contribution follows the Stokes constitutive relation $\tau = 2\mu S$, with μ the dynamic viscosity and $S = \frac{1}{2} (\nabla \vec{u} + (\nabla \vec{u})^T)$ the strain-rate tensor.

Additionally, the presence of an interface Γ splitting the domain into two regions imposes a discontinuity on the stress tensor. Surface tension acts as a momentum jump across the interface,

$$[\sigma] \hat{n} = -\gamma \kappa \hat{n}, \quad \text{equivalently} \quad [p] = \hat{n}^T [\tau] \hat{n} - \gamma \kappa, \quad (2)$$

where γ is the surface tension coefficient, assumed constant, κ the interface curvature and $\hat{n}$ the interface normal. This jump condition is the mechanism through which kinetic and surface energy are exchanged, and consequently the term whose discretization determines whether mechanical energy is preserved [1].

In order to capture the evolution of Γ without resorting to an explicit geometric reconstruction, the CLS approach represents the interface as an isosurface of a smooth marker function θ , built as a regularized Heaviside step function and advected by the incompressible velocity field,

$$\theta(r) = \frac{1}{2} \left(\tanh \frac{r}{2\varepsilon} + 1 \right), \quad \frac{\partial \theta}{\partial t} + (\vec{u} \cdot \nabla) \theta = 0, \quad (3)$$

where r is the signed shortest distance to the interface and ε is the smoothing factor controlling the interface thickness, such that $\theta(r) \rightarrow H(r)$ as $\varepsilon \rightarrow 0^+$. The geometric information required by equation (2) is then reconstructed from the marker itself,

$$\hat{n} = \frac{\nabla \theta}{|\nabla \theta|}, \quad \kappa = -\nabla \cdot \hat{n}. \quad (4)$$

The reader must bear in mind that the gradient of the marker function is, by construction, parallel to the interface normal; this property is the discrete hinge on which the energy-preserving curvature interpolation of Section 4.3 rests.

3 ADAPTIVE MOVING MESH STRATEGY

3.1 Why a hypoelastic variational mesher

The variational mesh generator of Garanzha and Kudryavtseva [13] relocates the mesh vertices so as to minimize a weighted polyconvex distortion functional built on quasi-isometric mappings. Four reasons motivate its adoption in the present framework.

First, the method is r -adaptive: topology and connectivity are fixed, nodes move, and control volumes are never added nor removed, so every algebraic operator retains its size and sparsity pattern throughout the run. Second, and as a direct consequence, each parallel rank keeps a fixed number of control volumes, so no load rebalancing is required and the scalability of the algebraic solver is preserved. Third, the hypoelastic stabilization is a stress-relaxation and continuation scheme rather than a plasticity model: at each step the reference metric is chosen so that the current mesh is stress-free, and the internal stresses are subsequently reintroduced in a gradual manner. The step-to-step deformation consequently tends to isometric as $\Delta t \rightarrow 0$, which delivers a continuous mesh update with no locked-in stresses and without solving a time-dependent mesh partial differential equation. Fourth, the adaptation is driven by a control (monitor) metric, so that the equidistribution principle provides a quasi-isometric quality bound and periodic remeshing becomes unnecessary.

It must be noted that the tracking of the interface is not an intrinsic feature of the variational algorithm. Garanzha and Kudryavtseva provide the deformation engine; the adaptation criterion is supplied here by feeding the level-set field into the control metric, so that the weight becomes large in a narrow band around the interface.

3.2 The Cartesian accordion restriction

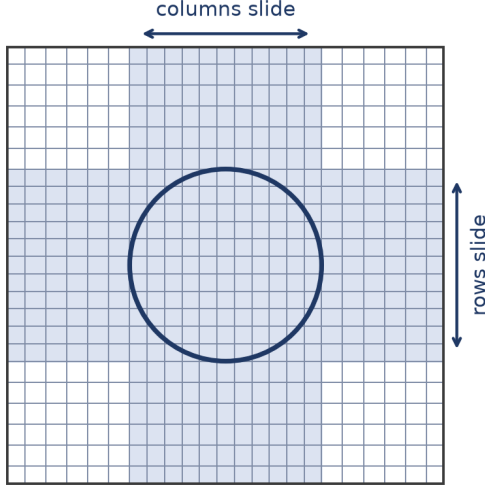
Furthermore, the variational principle has been adapted to the structured Cartesian solver in which the algebraic framework is implemented. Instead of solving for the position of every vertex, the unknowns are restricted to whole node lines that slide like a bellows, an arrangement referred to here as an *accordion* mesh, presented in Figure 1. Under this restriction the number of degrees of freedom of the deformation problem drops from $2(N + 1)(M + 1)$ to $(N + 1) + (M + 1)$ for an $N \times M$ mesh and, more importantly, the Newton system associated with each axis becomes tridiagonal, so the mesh update is solved in $\mathcal{O}(N + M)$ operations without recourse to an algebraic multigrid preconditioner.

Nevertheless, this efficiency is obtained at a price that must be stated plainly. Because only whole rows and columns are allowed to move, the refined region is not a compact patch around the interface but a band spanning the entire domain, as can be appreciated in Figure 1b. The restriction is consequently cheap in degrees of freedom and well suited to an interface that remains aligned with the mesh axes, whereas it becomes wasteful for a translating or strongly non-axis-aligned interface. Addressing this limitation without giving up the tridiagonal structure is an open point of the present development.

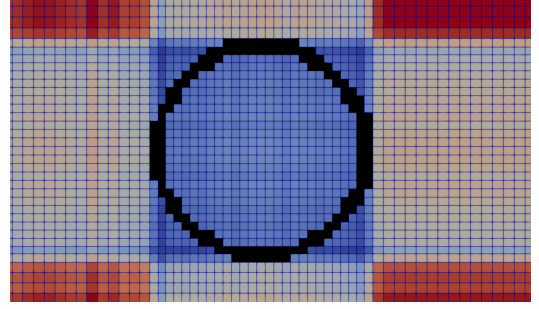
4 ALE FORMULATION AND THE GEOMETRIC CONSERVATION LAW

4.1 From the Reynolds transport theorem to the GCL

Once the mesh is allowed to move, the finite-volume balances must be written on a deforming control volume. Taking the material derivative of the integral of $\rho\phi$ over a control volume that



(a) Accordion restriction: whole rows and columns slide.



(b) Resulting adapted mesh around the interface.

Figure 1: Cartesian accordion adaptation driven by the level-set control metric

moves with velocity $\vec{w}$ yields the Reynolds transport theorem,

$$\frac{D}{Dt} \int_{\Omega_m} \rho \phi d\Omega = \frac{d}{dt} \int_{\Omega} \rho \phi d\Omega + \oint_{\partial\Omega} \rho \phi (\vec{v} - \vec{w}) \cdot \vec{n} ds, \quad (5)$$

where Ω_m denotes the material volume and Ω the control volume that moves with the mesh, $\vec{v}$ is the fluid velocity, $\vec{w}$ the mesh velocity, and the flux is therefore carried by the relative velocity $(\vec{v} - \vec{w})$. Setting $\phi = 1$ recovers mass conservation; for constant density and a solenoidal velocity field the fluid flux vanishes and only the mesh contribution survives, which delivers the continuous GCL,

$$\frac{d}{dt} \int_{\Omega} d\Omega = \oint_{\partial\Omega} \vec{w} \cdot \vec{n} ds = \int_{\Omega} \nabla \cdot \vec{w} d\Omega. \quad (6)$$

In words, the change of the volume of a cell must equal the volume swept by its faces. Equation (6) replicates the swept-volume GCL of Lesoinne and Farhat [16], recast here in the finite-volume, algebraic form employed throughout this work.

4.2 Algebraic semi-discrete system

The discretization follows the staggered, symmetry-preserving construction reviewed in [1], where the operators are assembled from incidence and metric matrices. The integrated divergence and the gradient read

$$\mathbf{M} = -\mathbf{T}_{FC} \mathbf{A}_F, \quad \mathbf{G} = (\Delta \mathbf{x}_F)^{-1} \mathbf{T}_{CF}, \quad \mathbf{G} = -\Omega_s^{-1} \mathbf{M}^T \Omega_c, \quad (7)$$

where $\mathbf{T}_{FC}$ is the face-to-cell incidence matrix and $\mathbf{T}_{CF} = \mathbf{T}_{FC}^T$, $\mathbf{A}_F$ and $\Delta \mathbf{x}_F$ are the diagonal arrangements of face areas and of the distances between cell centres, and Ω_c and Ω_s are the diagonal volume metrics of the cell and the staggered spaces, respectively. The third identity in (7) is the discrete duality between gradient and divergence, which is the property responsible for the null contribution of the pressure term to kinetic energy. For the reader coming from [1], note that $\mathbf{M}$ denotes here the *integrated* divergence, so that the divergence of that reference is recovered as $\Omega_c^{-1} \mathbf{M}$, and that its metrics $\mathbf{M}_C$, $\mathbf{M}_F$ are written here Ω_c , Ω_s .

On a moving mesh the semi-discrete system does not transport the intensive variables but the extensive ones, namely the staggered momentum $\Omega_s \mathbf{u}_s$ and the marker-weighted volume $\Omega_c \theta_c$, while the convective operators act on the ALE relative velocity,

$$\frac{d}{dt} (\Omega_s \mathbf{u}_s) + \mathbf{C}_s(\mathbf{u}_s - \mathbf{w}_s) \mathbf{u}_s = \mathbf{D} \mathbf{u}_s - \Omega_s \mathbf{G} p_c + \gamma \Omega_s \mathbf{K}_s \mathbf{G} \theta_c, \quad \mathbf{M} \mathbf{u}_s = 0, \quad (8)$$

$$\frac{d}{dt} (\Omega_c \theta_c) + \mathbf{C}_c(\mathbf{u}_s - \mathbf{w}_s) \theta_c = 0, \quad (9)$$

where $\mathbf{C}_s$ and $\mathbf{C}_c$ are the convective operators for momentum and marker, $\mathbf{w}_s$ is the staggered mesh velocity, $\mathbf{D}$ is the diffusive operator, p_c the cell-centred pressure and $\mathbf{K}_s = \text{diag}(\Upsilon \kappa_c)$ the staggered arrangement of the curvature. A purely Eulerian scheme is recovered for $\mathbf{w}_s = \mathbf{0}$, which is a useful consistency check on the implementation.

Therefore, since the metric moves with the mesh, $\Omega_k = \Omega_k(t)$, the time derivative of the extensive quantity expands as

$$\frac{d}{dt} (\Omega_k \phi_k) = \Omega_k \dot{\phi}_k + \dot{\Omega}_k \phi_k, \quad (10)$$

with $k = c$ for cell-centred and $k = s$ for staggered quantities. The additional term $\dot{\Omega}_k \phi_k$ is precisely the contribution that the GCL must reproduce, since $\dot{\Omega}_k$ has to equal the swept volume. It must be emphasized that Ω_k is the very same metric that supports the energy-preserving discretization; the geometric conservation law and the conservation of mechanical energy are consequently tied together through it, and they cannot be enforced independently.

4.3 Discrete energy preservation

The discrete kinetic and surface energies evolve through the same surface tension term, and the exchange between them is mediated by two interpolators,

$$\frac{d}{dt} E_k = \gamma (\mathbf{G} \theta_c)^T \mathbf{U}_s \Omega_s \Upsilon \kappa_c + \mathbf{u}_s^T \mathbf{D} \mathbf{u}_s, \quad \frac{d}{dt} E_p = \gamma (\mathbf{G} \theta_c)^T \mathbf{U}_s \Omega_s \Psi \kappa_c, \quad (11)$$

where $\mathbf{U}_s = \text{diag}(\mathbf{u}_s)$, κ_c is the cell-centred curvature, Υ is the interpolator used to shift the curvature from cells to faces and Ψ is the interpolator used in the advection of the marker. The viscous contribution $\mathbf{u}_s^T \mathbf{D} \mathbf{u}_s \leq 0$ is negative-definite and therefore dissipates kinetic energy, as expected.

The transfer between kinetic and surface energy is discretely exact if, and only if, the two interpolators are opposite. Splitting each of them into a symmetric part Π and a skew part Λ yields the energy-conserving condition

$$\Psi = \Pi + \Lambda \quad \implies \quad \Upsilon = \Pi - \Lambda, \quad (12)$$

so that an up-wind component in the advection of θ becomes a down-wind component in the curvature. This is the discrete counterpart of the first variation of area formula, and it is the property that the moving mesh must not destroy.

4.4 The discrete GCL and the midpoint integration rule

Since the swept-volume integrand is piecewise linear in time, the midpoint rule integrates the flux exactly provided that the mesh-velocity divergence is evaluated at $n + \frac{1}{2}$ and the transported variables are taken at their midpoint values,

$$\int_{t^n}^{t^{n+1}} \mathbf{F}^c dt = \Delta t \mathbf{F}^c \left(\text{Var}^{n+\frac{1}{2}}, \mathbf{Mw}^{n+\frac{1}{2}} \right). \quad (13)$$

This is the DGCL result of Farhat, Geuzaine and Grandmont [17] written in the present notation. Applying (13) to the volume and to the marker-weighted volume delivers the two conditions that the time integration must satisfy,

$$\Omega^{n+1} - \Omega^n = \Delta t \mathbf{Mw}^{n+\frac{1}{2}}, \quad \Omega^{n+1} \theta^{n+1} - \Omega^n \theta^n = \Delta t \theta^{n+\frac{1}{2}} \mathbf{Mw}^{n+\frac{1}{2}}, \quad (14)$$

where $\mathbf{Mw}$ is the integrated divergence of the mesh velocity, i.e. the swept volume per unit time. The relevance of (14) is not merely formal: the DGCL has been shown to be a necessary and sufficient condition for an ALE scheme to inherit the nonlinear stability of its fixed-grid counterpart [17].

4.5 Time integration strategy

Nevertheless, a conflict arises when (14) is combined with the incompressibility constraint. The DGCL requires the mesh velocity at the midpoint $n + \frac{1}{2}$, whereas the fractional step method imposes the discrete divergence constraint on the mesh configuration at $n + 1$. In order to reconcile both requirements, the extensive quantities are advanced with a second-order Adams–Bashforth (AB2) scheme,

$$(\Omega\phi)^{n+1} = (\Omega\phi)^n + \Delta t \left(\frac{3}{2} \mathbf{R}^n - \frac{1}{2} \mathbf{R}^{n-1} \right), \quad \mathbf{M}^{n+1} \mathbf{u} = 0, \quad \mathbf{R} = \mathbf{R}(\text{ops}^{n+1}, \text{vars}^n), \quad (15)$$

where the residual $\mathbf{R}$ is assembled with the operators and the geometry evaluated on the new mesh configuration while the transported variables remain lagged at n , a choice referred to here as a lagged-mesh strategy since the mesh trails the flow. In practice, each time step comprises: (i) deformation of the mesh from the marker field θ^n , with the nodal displacement sub-relaxed

Table 1: Parameters of the reported test cases

Parameter	Free-stream test	Cylindrical column
Domain $[0, 2H]^2, H$	0.5	0.5
Grid $N \times M$	64×64	128×128
Interface radius R_0	0.15	$0.15 = 0.3H$
Interface thickness ε (cells)	1.5	1.5
Density ρ (both phases)	1.0	1.0
Perturbation mode s	2	2
Time step Δt	10^{-3} (fixed)	dynamic, CFL-limited
Marker advection scheme	SUPERBEE	SUPERBEE
Time integrator	AB2	AB2

so that the mesh Courant number stays bounded, which yields $\mathbf{w}_s$; (ii) reconstruction of the operators and interpolators on the mesh at $n+1$ using the relative velocity $\mathbf{u}_s^n - \mathbf{w}_s$; (iii) an AB2 predictor for the extensive momentum; (iv) the Poisson solve enforcing $\mathbf{M}^{n+1}\mathbf{u} = 0$ and the subsequent velocity correction; and (v) an AB2 update of the marker-weighted volume followed by the recovery of $\theta^{n+1} = (\Omega_c \theta_c)^{n+1} / \Omega_c^{n+1}$. Among the integrators tested, this combination has empirically delivered the lowest DGCL error and the most robust behaviour under mesh motion.

5 RESULTS

Following the practice of [1], the simulations reported here are inviscid and the density ratio is fixed to unity, in order to isolate the behaviour of the surface tension term and of the mesh motion from the additional complications introduced by variable properties. The general formulation of Section 2 is nonetheless retained, since the extension to variable properties does not modify the geometric argument developed in Section 4. Table 1 holds the parameters of the cases discussed below.

The reference scales are those provided by linear perturbation theory for an initially cylindrical interface perturbed as $r(\vartheta) = R_0 + r_p \cos(s\vartheta)$, namely

$$T = 2\pi \sqrt{\frac{2\rho R_0^3}{\gamma s(s^2 - 1)}}, \quad c = \sqrt{\frac{\gamma s(s^2 - 1)}{2\rho R_0}}, \quad (16)$$

where T is the oscillation period and c the characteristic speed. It must be noted that linear theory assumes a small interfacial deformation, a condition that the present cases do not necessarily satisfy; the expressions above are consequently used as reference scales rather than as an analytical solution.

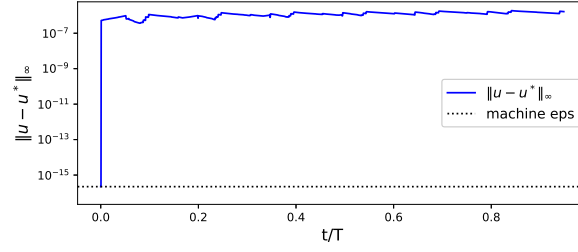


Figure 2: Free-stream error $\|\mathbf{u} - \mathbf{u}^*\|_\infty$ on a moving mesh (DGCL test)

5.1 Free-stream preservation

The first assessment is a pure diagnostic. A uniform diagonal velocity field $\mathbf{u}^*$ is advected on a fully periodic, moving mesh whose motion is still driven by the marker, so that the ALE terms are fully exercised while the surface tension forcing is removed. Under these conditions the uniform field is an exact solution and the only mechanism able to perturb it is the DGCL error; if the scheme respects the DGCL, $\|\mathbf{u} - \mathbf{u}^*\|_\infty$ must remain at round-off.

Figure 2 presents the evolution of this error. The result is not the expected one and is reported as observed: the error departs from machine precision within roughly the first hundred iterations, climbs to approximately 10^{-7} and subsequently plateaus, remaining of that order for the rest of the integration. Therefore, the present time-integration strategy does not satisfy the DGCL exactly. Following the analysis of [17], this implies that the scheme cannot be guaranteed to inherit the nonlinear stability of its fixed-grid counterpart, which identifies a credible root cause for the instability observed under large mesh deformation.

5.2 Cylindrical column benchmark

The canonical benchmark is the zero-gravity cylindrical column of [1], in which a liquid column of radius $R_0 = 0.3H$ is placed at the centre of the domain with the velocity field initially at rest. In the absence of numerical error the fluid should remain stagnant; in practice, spurious currents appear as a consequence of the imbalance between the discrete curvature and the pressure gradient, so the case measures directly the quality of the surface tension discretization. Energy levels are scaled by, and referenced to, the initial mechanical energy, and the integration is carried out over five oscillation periods.

Figure 3 presents the energy balance for the four combinations of interest, obtained by enabling or disabling the opposite-interpolator condition (12) and by running on a static or on an adaptive moving mesh. When the energy-preserving condition holds and the mesh is static, Figure 3a, the numerical growth of kinetic energy is counterbalanced by an equal decrease of surface energy: both excursions peak at 7.7×10^{-6} of the reference energy around $t/T \approx 0.35$ and decay back to their initial state, while mechanical energy remains bounded by 5×10^{-8} throughout the five periods. This reproduces the behaviour reported in [1]. Conversely, when the condition is violated, Figure 3c, surface energy grows monotonically and mechanical energy is no longer conserved, reaching 2.4×10^{-4} by $t/T = 5$.

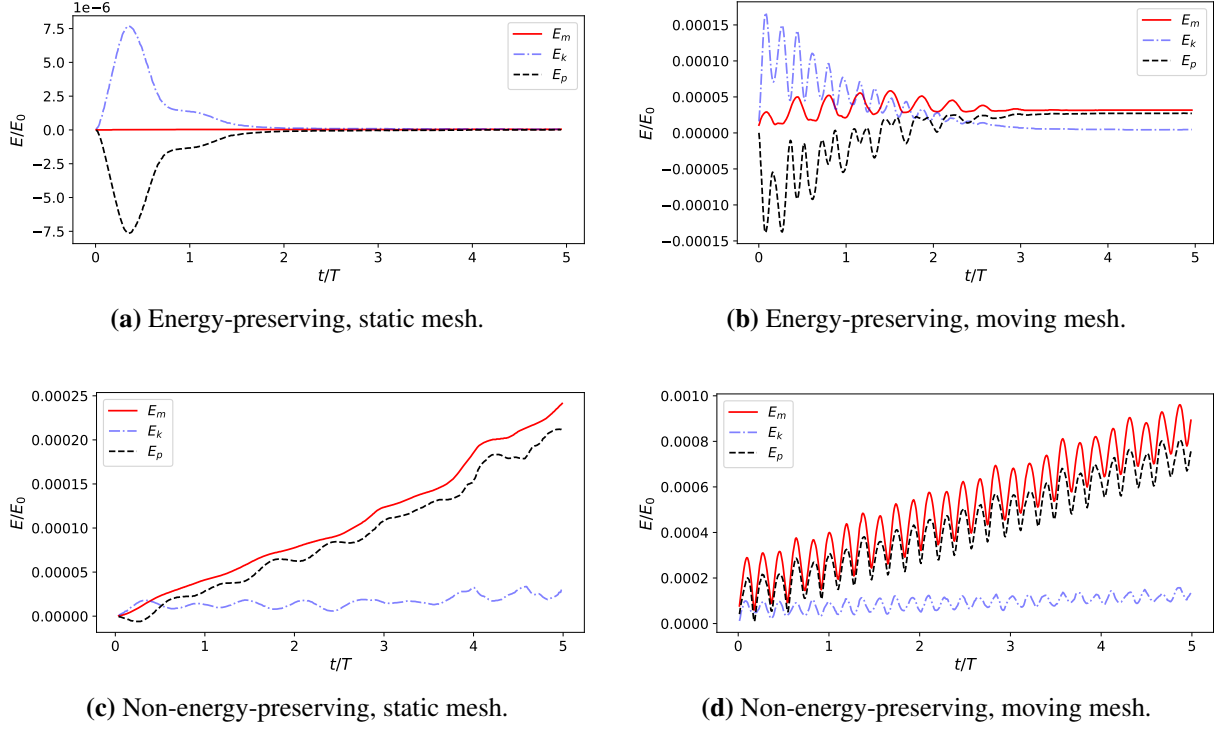


Figure 3: Energy balance for the cylindrical column on a 128×128 grid. E_k kinetic, E_p surface and E_m mechanical energy, referenced to the initial mechanical energy

Moving the mesh degrades this picture, though less severely than the static-frame analysis might suggest. With the energy-preserving condition active, Figure 3b, the balance is not closed: mechanical energy overshoots to 5.8×10^{-5} during the first period and then settles on a plateau of 3.2×10^{-5} . Nevertheless, the departure is bounded; the oscillation amplitude of E_m falls from 4.1×10^{-5} over the first period to 6.7×10^{-7} beyond $t/T = 3$, so the exchange between kinetic and surface energy damps out and the solution reaches a steady state instead of drifting. When the energy-preserving condition is dropped and the mesh is moved, Figure 3d, the monotonic growth of the static case reappears superimposed on the oscillation and mechanical energy reaches 9.6×10^{-4} , a factor of thirty above the plateau of its conservative counterpart.

Therefore, two observations can be extracted. On the one hand, the energy-preserving discretization retains its qualitative advantage on a moving mesh: the conservative variant is consistently better behaved than the non-conservative one in both frames, so the mesh motion does not cancel the benefit of the mimetic curvature interpolation. On the other hand, the mesh motion does introduce an energy imbalance that the static-frame analysis does not show, and whose magnitude grows with the mesh distortion imposed per time step. However, This is consistent with the free-stream diagnostic of Section 5.1: in both cases the error accumulates over the initial transient and then saturates rather than growing without bound. Behaving as a bounded

spurious source in the discrete energy balance, which is what keeps the scheme usable despite not satisfying the DGCL exactly.

6 CONCLUSIONS AND FUTURE WORK

A framework coupling a conservative, algebraic, energy-preserving level-set discretization with a hypoelastic variational moving mesh has been presented. The GCL has been cast algebraically and made consistent with the very same volume metric Ω that supports the energy-preserving scheme, and the swept volumes have been evaluated geometrically so that the geometric side of the constraint closes to machine precision by construction. On the mesh side, the Cartesian accordion restriction reduces the variational deformation problem to a tridiagonal system per axis while the level-set field drives the control metric, which preserves both the sparsity of the algebraic operators and the parallel load balance.

The assessment delivers a mixed but informative picture. The moving-mesh solver is second-order accurate in time, and the energy-preserving curvature interpolation retains its advantage over the standard one when the mesh is deformed. Nevertheless, mesh deformation does affect the conservation of mechanical energy, and the instabilities grow when the mesh distortion imposed per step becomes too large. The free-stream test isolates the cause: the proposed AB2 lagged-mesh scheme provides the lowest DGCL error among the alternatives tested, yet it does not preserve the free stream to machine precision, and consequently the nonlinear stability guarantees of the fixed-grid scheme cannot be claimed for the coupled framework as it currently stands.

Future work is therefore concentrated on the temporal integration. An implicit Crank–Nicolson scheme evaluates the quadrature at the midpoint naturally, which aligns it with the requirement of equation (13); a DGCL-consistent energy-balance study along these lines is presently under way. In parallel, the spurious currents arising from the momentum-preservation step are being addressed, since these originate in the mixing of temporal states, namely operators evaluated at $n + 1$ acting on variables lagged at n . Extending the accordion restriction so that refinement remains local for a translating interface, and generalizing the analysis to variable density and viscosity, complete the roadmap of this development.

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