

Adaptive Moving Mesh for Conservative Multiphase Flows Discretization

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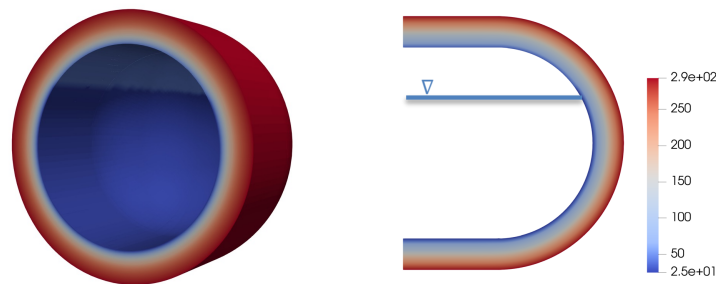
Preliminary results — work in progress

Motivation — the gap

Motivation · Adaptive mesh · Numerical method · Results · Outlook

Coupling multiphase solvers with mesh adaptation is not new — the question is how it blends with algebraic methods.

- To develop a coupled multiphase solver with adaptive mesh refinement for free surface and falling film tracking.
- Many AMR / moving-mesh couplings for multiphase flow already exist.
- Adapting a methodology that best blends with algebraic operators...
 - ...without sacrificing energy preserving, portability and scalability on heterogeneous HPC.
- So, the mesh-motion strategy must respect that same algebraic, operator-level structure.

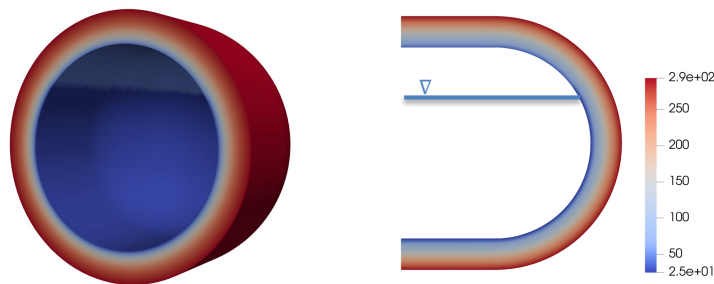


Motivation — research questions

Motivation · Adaptive mesh · Numerical method · Results · Outlook

Does mesh motion preserve the guarantees the numerical method was built to provide?

- Does the coupled scheme remain a conservative framework?
- How much mesh motion impact analyses that are valid on a static frame?
 - Energy-preservation proofs assume a fixed mesh — what survives when the mesh moves?



Adaptive mesh — why a hypoelastic variational mesher

Motivation · **Adaptive mesh** · Numerical method · Results · Outlook

We adopt Garanzha & Kudryavtseva's variational quasi-isometric mesher with hypoelastic stabilization. Four reasons:

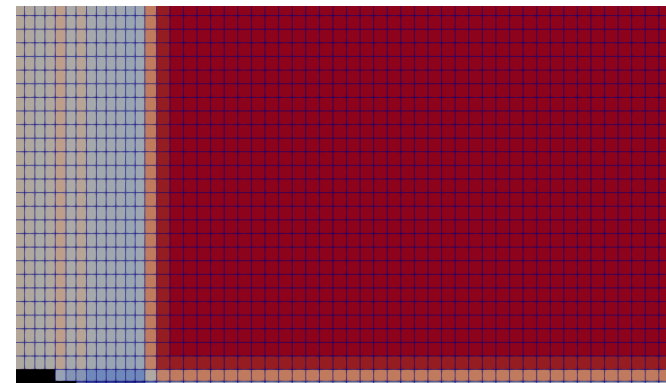
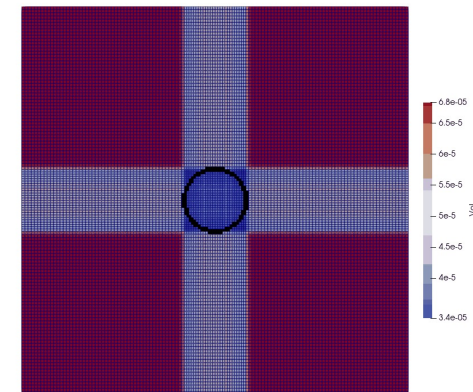
- r-adaptive (node-relocation) mesh — fixed topology/connectivity; nodes move, control volumes are never added or removed.
 - Matrices keep their size, sparsity and domain throughout the run.
- Consequently: no MPI load imbalance — each rank keeps a fixed number of control volumes.
 - Preserves scalability and portability of the algebraic solver.
- Hypoelastic stabilization: each incremental deformation is made stress-free, then stress is reintroduced gradually.
 - No locked-in mesh stresses; steps stay quasi-isometric; no mesh PDE is solved.
- Adaptation driven by a control (monitor) metric — the equidistribution principle guarantees mesh quality.

Adaptive mesh — our Cartesian 'accordion' adaptation

Motivation · **Adaptive mesh** · Numerical method · Results · Outlook

We adapt the variational principle to a structured Cartesian solver.

- 'Accordion' mesh: whole node lines slide like a bellows.
- Solve for the rows and columns only — not for every vertex.
 - Large drop in degrees of freedom vs. the full unstructured variational problem.
- Control metric refines near the interface; we drive it with the level-set field (our coupling).
 - Quasi-isometric functional keeps mesh quality — no remeshing.
 - Trade-off: refinement spans a band across the domain (see figure).



Numerical method — continuum (conservative level-set)

Motivation · Adaptive mesh · **Numerical method** · Results · Outlook

Incompressible Navier–Stokes + a conservative level-set marker (Valle, Trias & Castro, 2020).

- Marker θ advected by an incompressible field; normal and curvature reconstructed from θ .
- Surface tension enters as a momentum jump across the interface.

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = -\nabla p + \nabla \cdot \tau, \quad \nabla \cdot \vec{u} = 0$$

incompressible Navier–Stokes

$$\theta(r) = \frac{1}{2} \left(\tanh \frac{r}{2\varepsilon} + 1 \right), \quad \frac{\partial \theta}{\partial t} + (\vec{u} \cdot \nabla) \theta = 0$$

smoothed marker & its advection

$$\hat{n} = \frac{\nabla \theta}{|\nabla \theta|}, \quad \kappa = -\nabla \cdot \hat{n}, \quad [\sigma] \hat{n} = -\gamma \kappa \hat{n}$$

normal, curvature and surface-tension jump

Geometric Conservation Law (GCL)

Motivation · Adaptive mesh · **Numerical method** · Results · Outlook

Start from the Reynolds Transport Theorem; mass conservation on a moving control volume yields the GCL.

- GCL: cell-volume change = volume swept by the mesh velocity $\mathbf{w}$ — following Lesoinne & Farhat (1996).

$$\frac{D}{Dt} \int_{\Omega_s} \rho \phi \, dv = \frac{d}{dt} \int_{\Omega} \rho \phi \, dv + \oint_{\partial\Omega} \rho \phi (\vec{v} - \vec{w}) \cdot \vec{n} \, ds$$

Reynolds Transport Theorem (mesh velocity $\mathbf{w}$)

$$\frac{d}{dt} \int_{\Omega} d\Omega = \oint_{\partial\Omega} \vec{w} \cdot \vec{n} \, ds = \int_{\Omega} \nabla \cdot \vec{w} \, d\Omega$$

continuous GCL: volume change = swept volume

$$\Omega^{n+1} - \Omega^n = \Delta t \mathbf{M} \mathbf{w}$$

discrete GCL (integrated over the step)

GCL — the semi-discrete system transports $\Omega \cdot \mathbf{u}$, not $\mathbf{u}$

Motivation · Adaptive mesh · **Numerical method** · Results · Outlook

The time derivative acts on the extensive quantity (momentum / marker-volume) — not on the intensive variable alone.

- On a moving mesh $\Omega_k = \Omega_k(t)$, so $d/dt(\Omega_k \phi_k) = \Omega_k \dot{\phi}_k + \dot{\Omega}_k \phi_k$.
 - The extra term $\dot{\Omega}_k \phi_k$ is exactly the GCL contribution ($\dot{\Omega}_k = \text{swept volume}$).
- Same metric Ω_k as the energy-preserving scheme $\rightarrow$ GCL and energy conservation share it.

$$\frac{d}{dt}(\Omega_s \mathbf{u}_s) + \mathbf{C}_s(\mathbf{u}_s - \mathbf{w}_s)\mathbf{u}_s = \mathbf{D}\mathbf{u}_s - \Omega_s \mathbf{G}p_c + \gamma \Omega_s \mathbf{K}_s \mathbf{G}\theta_c, \quad \mathbf{M}\mathbf{u}_s = 0$$

staggered momentum (diffusion – pressure + surface tension), incompressible

$$\frac{d}{dt}(\Omega_c \theta_c) + \mathbf{C}_c(\mathbf{u}_s - \mathbf{w}_s)\theta_c = 0$$

marker (level-set) advection at cells

$$\frac{d}{dt}(\Omega_k \phi_k) = \Omega_k \dot{\phi}_k + \dot{\Omega}_k \phi_k$$

$\dot{\Omega}_k \phi_k$ is the GCL term — $\dot{\Omega}_k$ must equal the swept volume

Numerical method — energy preservation

Motivation · Adaptive mesh · **Numerical method** · Results · Outlook

Kinetic and surface energy exchange through the surface-tension term — the transfer must be exact at the discrete level.

- Discrete kinetic and surface energy evolve through the same surface-tension term, with curvature interpolators Υ and Ψ .
- Energy-conserving condition: Υ is the opposite of Ψ (they differ only in the sign of the upwind part Λ).
 - An up-wind component in the advection of θ becomes a down-wind component in the curvature $\rightarrow$ exchange is discretely exact.

$$\frac{d}{dt} E_{k_s} = \gamma (\mathbf{G}\theta_c)^T \mathbf{U}_s \Omega_s \Upsilon \kappa_c + \mathbf{u}_s^T \mathbf{D} \mathbf{u}_s$$

discrete kinetic energy (Υ curvature interpolator; $\mathbf{u}_s^T \mathbf{D} \mathbf{u}_s \leq 0$ viscous)

$$\frac{d}{dt} E_{p_s} = \gamma (\mathbf{G}\theta_c)^T \mathbf{U}_s \Omega_s \Psi \kappa_c$$

discrete surface (potential) energy (Ψ advection interpolator)

$$\Psi = \Pi + \Lambda \quad \Rightarrow \quad \Upsilon = \Pi - \Lambda$$

opposite interpolators $\Rightarrow$ exact kinetic $\leftrightarrow$ surface transfer

GCL — Farhat's integration rule

Motivation · Adaptive mesh · **Numerical method** · Results · Outlook

To satisfy the discrete GCL, integrate the fluxes with the same temporal quadrature, at the midpoint of the values.

- Fluxes evaluated at midpoint values; the mesh-velocity divergence $\mathbf{Mw}$ enters at $n+\frac{1}{2}$. (following Farhat, Geuzaine & Grandmont, 2001).
- GCL on the volume, and on the marker-weighted volume $\Omega \theta$.

$$\int_{t^n}^{t^{n+1}} \mathbf{F}^c dt = \Delta t \mathbf{F}^c(\mathbf{Var}^{n+\frac{1}{2}}, \mathbf{Mw}^{n+\frac{1}{2}})$$

midpoint temporal quadrature ($\mathbf{Mw}$ = mesh-velocity divergence)

$$\Omega^{n+1} - \Omega^n = \Delta t \mathbf{Mw}^{n+\frac{1}{2}}$$

discrete GCL on the volume

$$\Omega^{n+1} \theta^{n+1} - \Omega^n \theta^n = \Delta t \theta^{n+\frac{1}{2}} \mathbf{Mw}^{n+\frac{1}{2}}$$

GCL on the marker-weighted volume $\Omega \theta$

GCL — our stabilization strategy

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Choosing $w^{n+\frac{1}{2}}$ together with the divergence evaluated at $n+1$ fixes the scheme.

- Constraint: mesh velocity at the midpoint, but the discrete divergence taken at $n+1$.
- Our stable choice: an Adams–Bashforth (AB2) update with
 - geometry / operators evaluated at $n+1$,
 - transported variables lagged at n ,
 - a lagged-mesh strategy (mesh trails the flow).

$$(\Omega\phi)^{n+1} = (\Omega\phi)^n + \Delta t \left(\frac{3}{2} \mathbf{R}^n - \frac{1}{2} \mathbf{R}^{n-1} \right)$$

AB2 update of the extensive quantity $\Omega\phi$

$$\mathbf{M}^{n+1} \mathbf{u} = 0, \quad \mathbf{R} = \mathbf{R}(\text{ops}^{n+1}, \text{vars}^n)$$

divergence & operators at $n+1$, variables lagged at n

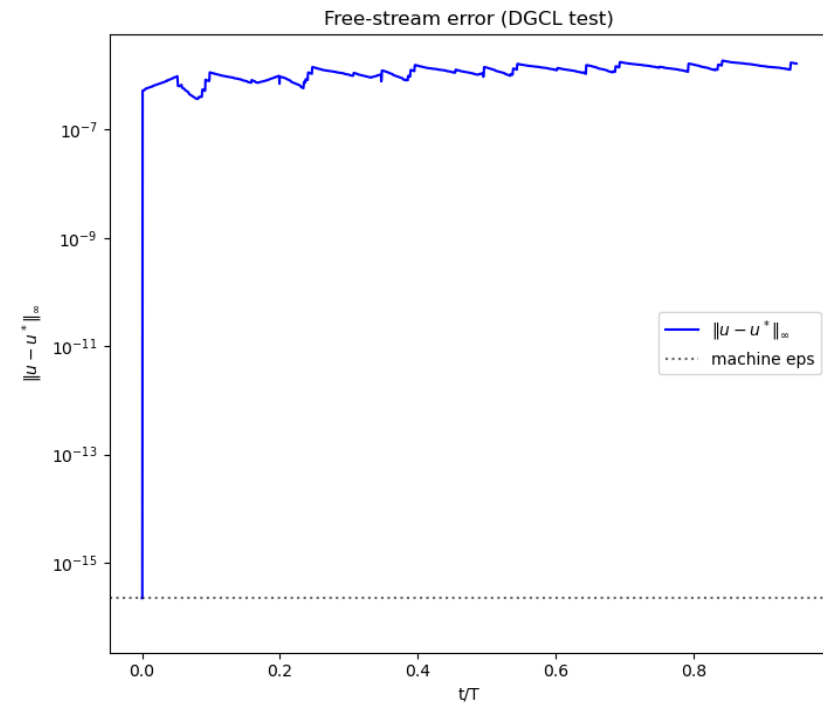
Preliminary Results

Results — free-stream preservation (DGCL test)

Motivation · Adaptive mesh · Numerical method · **Results** · Outlook

is it kept constant to machine precision?

- Error grows
 - Farhat, Geuzaine & Grandmont (2001): DGCL is necessary & sufficient to inherit the fixed-grid nonlinear stability.
- Isolates the cause of the large-deformation instability:
 - DGCL violation (time integration) vs. mesh-quality degradation (skewness / aspect ratio).

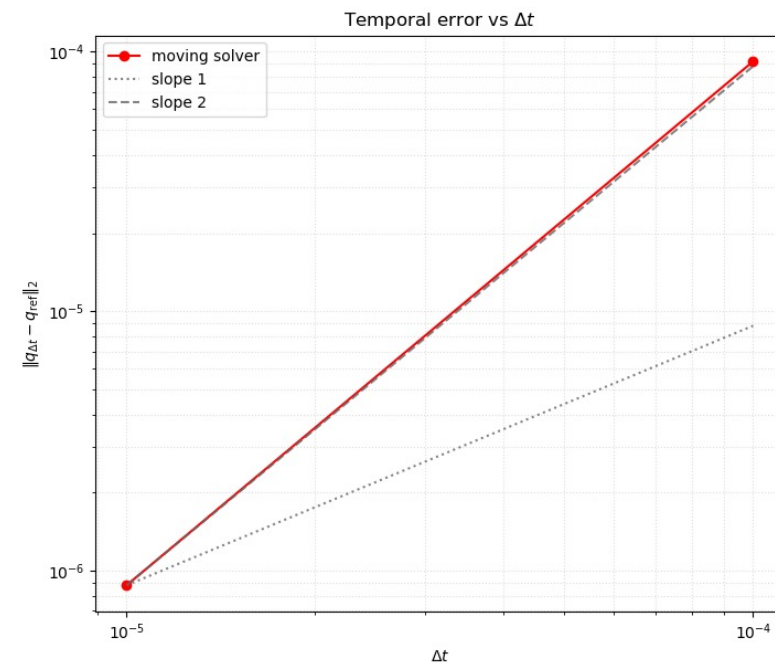


Results — free-stream preservation (DGCL test)

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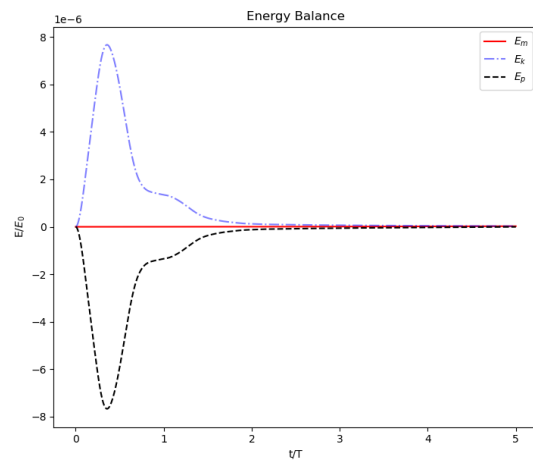


Results — canonical benchmark & energy transfer

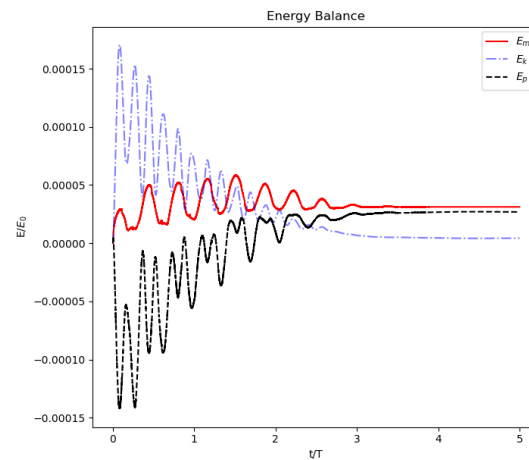
Motivation · Adaptive mesh · Numerical method · **Results** · Outlook

Static bubble benchmark – Energy preserving method on a 128x128 grid

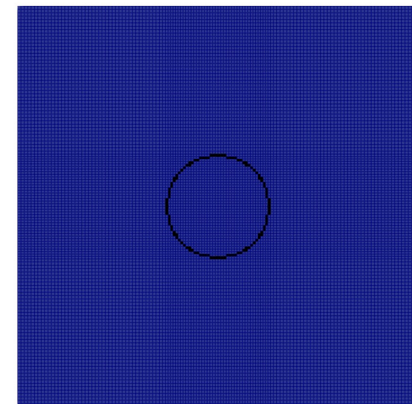
Static mesh



Moving mesh



Velocity profile



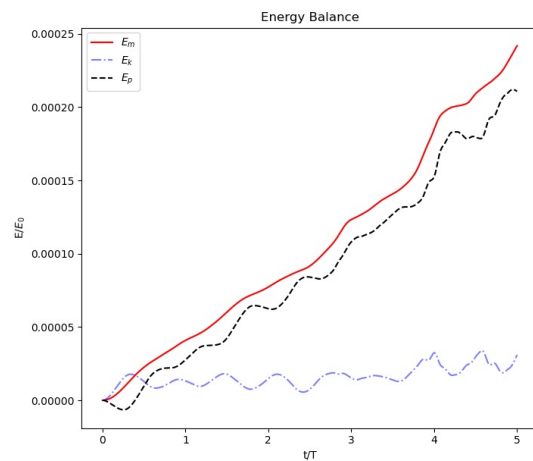
dt reduction from $1e-3$ to $1e-5$

Results — canonical benchmark & energy transfer

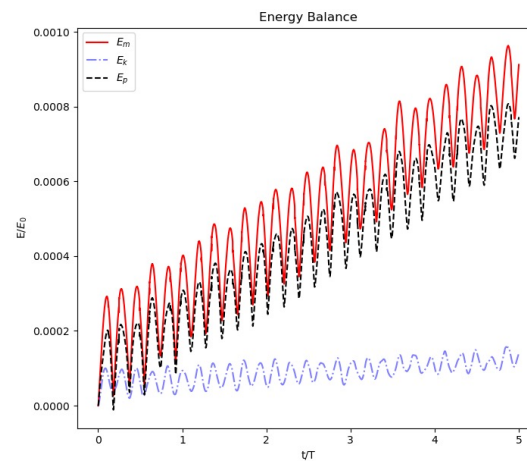
Motivation · Adaptive mesh · Numerical method · **Results** · Outlook

Static bubble benchmark – Not energy preserving method on a 128x128 grid

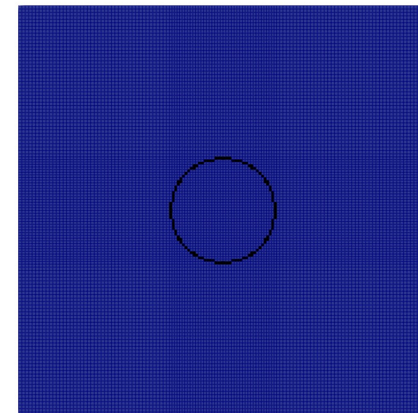
Static mesh



Moving mesh



Velocity profile



Conclusions (preliminary)

Motivation · Adaptive mesh · Numerical method · Results · **Outlook**

- A framework coupling a conservative algebraic energy-preserving level-set with a hypoelastic moving mesh.
 - The GCL, cast algebraically — consistent with the metric Ω of the energy-preserving scheme.
- Mesh deformation affects energy preservation during instability face / high spurious currents scenario.
 - Instabilities grow when the mesh distortion per step is too high.
- The AB2 scheme (geometry $n+1$ / variables n / lagged mesh) provides the lowest DGCL error.
 - Higher efforts to improve time integration scheme are required.
- Additional effort is required to pursue a DGCL preservation and improve time integration.

Outlook — future work

Motivation · Adaptive mesh · Numerical method · Results · **Outlook**

- Implicit Crank–Nicolson time integration.
 - Naturally evaluates the temporal quadrature at the midpoint — aligned with the DGCL requirement.
- Reduce the spurious currents from momentum preservation.
 - They arise from mixing temporal states: operators at $n+1$ acting on variables at n .

Thank you

Questions?