

Minimum-dissipation model and symmetry-preserving discretization for turbulent thermal convection

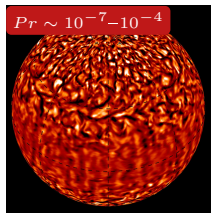
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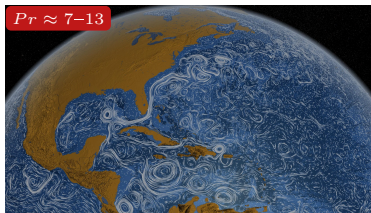
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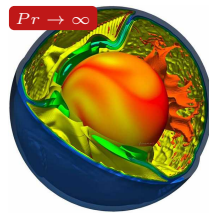
Turbulent Thermal Convection



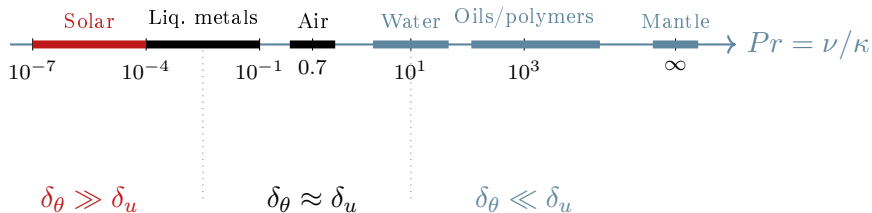
Solar convection zone



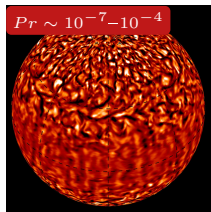
Ocean currents



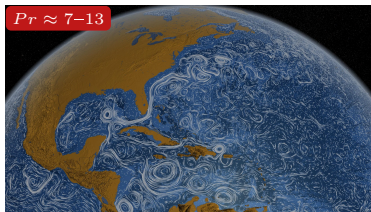
Earth mantle convection



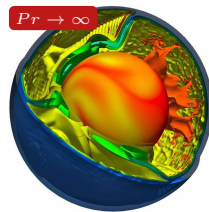
Turbulent Thermal Convection



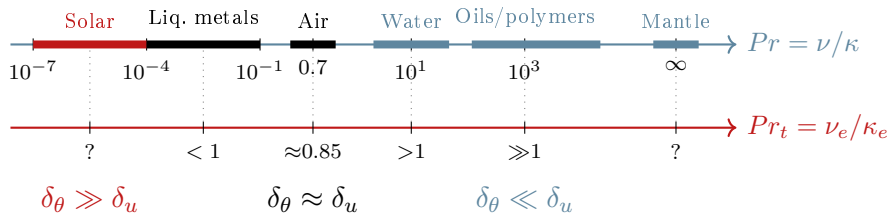
Solar convection zone



Ocean currents



Earth mantle convection



① Numerical method

② Test case: Differentially Heated Cavity

③ Test case: Rayleigh–Bénard Convection

Governing Equation

$$\partial_t u + \nabla \cdot (uu) - \nabla(\nu \nabla u) = -\nabla p + \mathbf{g}\beta(T - T_0) + f_c \epsilon_{ij3} u,$$

$$\partial_t T + \nabla \cdot (uT) - \nabla(\kappa \nabla T) = 0,$$

$$\nabla \cdot u = 0$$

Skew-Symmetry of Convective Operator

$$(u \cdot \nabla)^* = -(u \cdot \nabla)$$

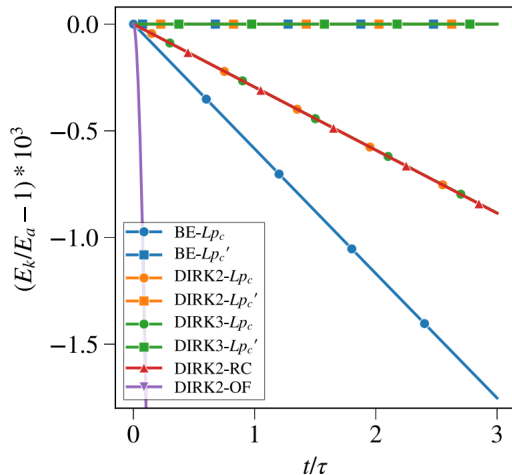
Symmetry of Diffusive Operator

$$\nabla^* = -\nabla \cdot \quad \Rightarrow \quad (\nabla \cdot \nabla)^* = (\nabla \cdot \nabla)$$

Discrete Operators

$$\begin{aligned}\phi_{i-\frac{1}{2}} &= \frac{1}{2}(\phi_{i-1} + \phi_i), \\ [C_c(\mathbf{u}_s)]_{k,k} &= \frac{1}{2} \sum_{f \in F_f(k)} [\mathbf{u}_s]_f A_f = 0\end{aligned}$$

- Inviscid Taylor-Green vortex
- Lp_c : Chorin
 Lp'_c : van Kan
 RC : Rhie-Chow
 OF : OpenFOAM
rest: Ed Komen et al



[1]E.M.J. Komen, J.A. Hopman, E.M.A. Frederix, F.X. Trias, R.W.C.P. Verstappen, Computers & Fluids, Volume 225, 2021, 104979.

Spatial filter

- $\bar{u} = \frac{1}{|\Omega_\Delta|} \int_{\Omega_\Delta} u(x, t) dx$

Spatial filter

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$$\partial_t \bar{T} + \nabla \cdot (\bar{u} \bar{T}) - \nabla (\kappa \nabla \bar{T}) = - \underbrace{\nabla \cdot (\overline{uT} - \bar{u} \bar{T})}_{\text{subgrid heat flux}},$$

$$\nabla \cdot \bar{u} = 0$$

Spatial filter

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$$\nabla \cdot \bar{u} = 0$$

Closure Model

- $$-2\nu_e S(\bar{u}) \approx \overline{uu^T} - \bar{u} \bar{u}^T$$

Poincaré Inequality $\implies$ unsolved scale v' satisfy

$$\int_{\Omega_{\Delta}} \frac{1}{2} \|v'\|^2 dx \leq C_{\Delta} \int_{\Omega_{\Delta}} \frac{1}{2} \|\nabla v\|^2 dx$$

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Dissipation Condition

$$\frac{d}{dt} \int_{\Omega_\Delta} \frac{1}{2} \|\nabla v\|^2 dx = -\nu \int_{\Omega_\Delta} \|\nabla^2 v\|^2 dx$$

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Eddy Viscosity

$$\nu_e = C_{\Delta} \frac{|\overline{r(v)} + \frac{1}{4}g\beta \nabla v_g \cdot \nabla (\bar{T} - \bar{T}_0)|}{\overline{q(v)}}$$

Poincaré Inequality $\implies$ unsolved scale T' satisfy

$$\int_{\Omega_{\Delta}} \|T'\|^2 dx \leq C_{\Delta} \int_{\Omega_{\Delta}} \|\nabla \bar{T}\|^2 dx$$

Potential Temperature Gradient Intensity

$$\frac{d}{dt} \int_{\Omega_{\Delta}} \frac{1}{2} \|\nabla \bar{T}\|^2 dx = -\kappa \int_{\Omega_{\Delta}} \|\nabla^2 \bar{T}\|^2 dx$$

Eddy Diffusivity

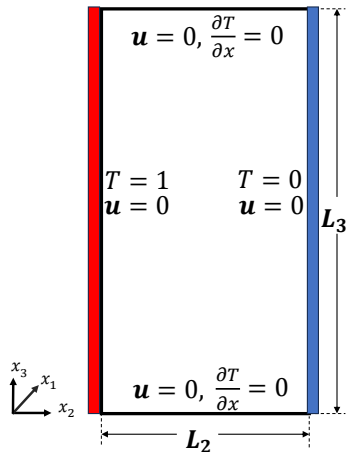
$$\kappa_e = \frac{-(\nabla v \nabla \bar{T}) \cdot \nabla \bar{T}}{\|\nabla \bar{T}\|^2}$$

- Non-constant Pr_t
- Include **buoyancy** effect
- Include **stratification** effect
- Switch between stable/unstable and neutral stratification
- Switch off in **Laminar** and **Transitional** flow
- Switch off at a No-slip wall
- Open Source

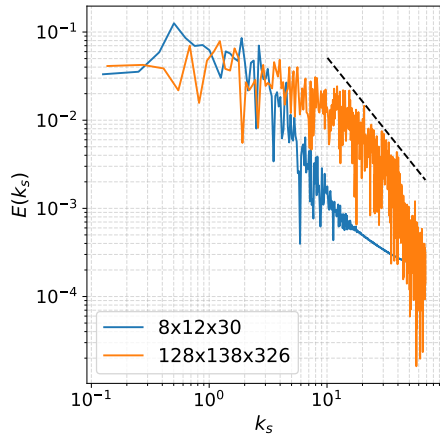
https://github.com/teresajingsun/Minimum_Dissipation_Model.git

- ① Numerical method
- ② Test case: Differentially Heated Cavity
- ③ Test case: Rayleigh–Bénard Convection

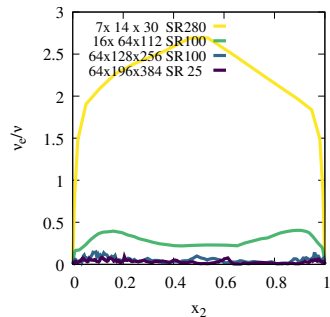
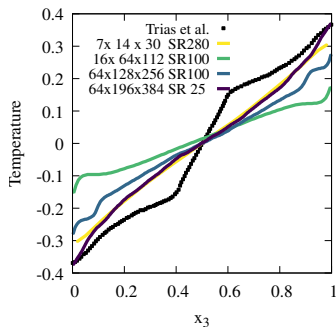
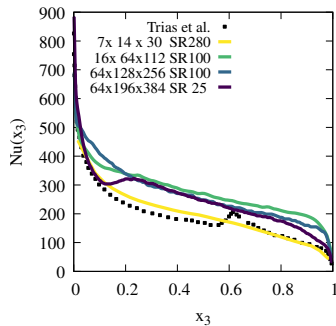
Setup of Differentially Heated Cavity



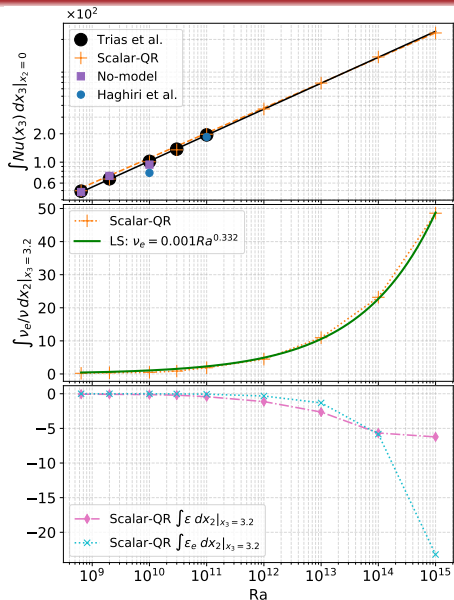
- $Ra = \frac{g\beta\Delta TL_3^3}{\nu\kappa}$
 $6.4 \times 10^8 \leq Ra \leq 10^{15}$
- $Pr = 0.71$
- Height Aspect Ratio $L_3/L_2 = 4$
- Periodic Boundary Condition in x_1
- $Pr_t = 1$
- $\nu_e, C = 0.024$



Kolmogorov scaling: $E(\kappa_s) = C \varepsilon^{2/3} \kappa_s^{-5/3}$



Convergence: $7 \times 14 \times 30$

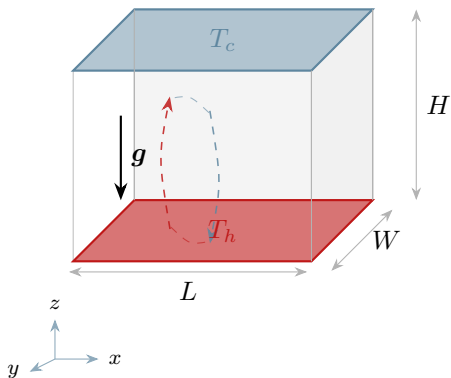


- $Nu_{scalar-QR} = 0.157Ra^{0.281}$

- $Nu_{DNS} = 0.182Ra^{0.275}$.

- Relative error $\leq 1.6\%$

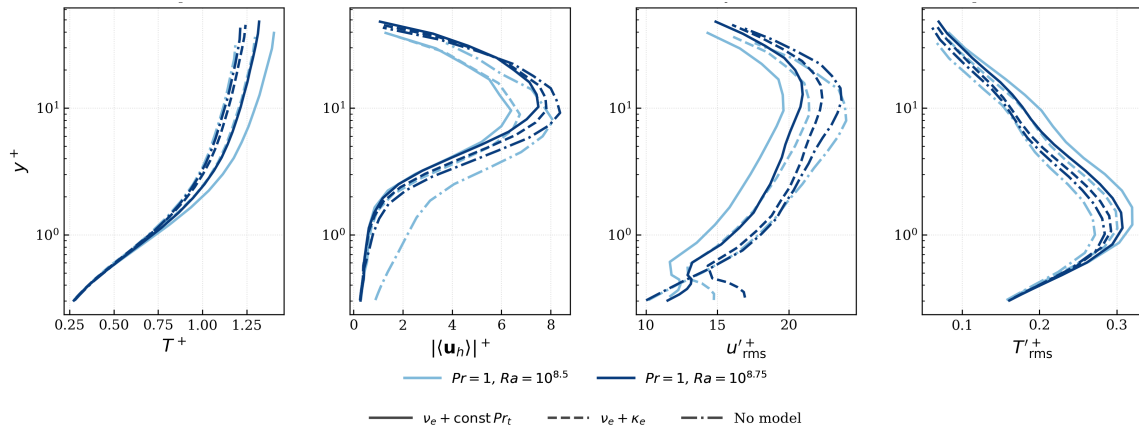
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- $Pr = \frac{\nu}{\kappa}$
- $Ra = \frac{g \beta \Delta T H^3}{\nu \kappa}$
- $L/H = 1$
- ν_e and κ_e , $C = 0.024$

Pr	Ra	Model	Nu	Nu_h	Nu_{ref}	Max. diff.	$Nu_{\epsilon_T}/Nu_{\text{ref}}-1$	Δ_{max}/η	t_{avg}/t_f
1	e7	$\nu_e + \kappa_e$	17.75	17.75	17.16	3.5%	-10.5%	2	2580
		$\nu_e + \text{const } Pr_t$	17.90	17.76	17.16	4.3%	-11.8%	2	1091
		No model	—	—	—	—	—	—	—
1	e8	$\nu_e + \kappa_e$	30.75 [†]	30.73 [~]	32.85	6.4%	-26.3%	50	2591
		$\nu_e + \text{const } Pr_t$	36.26	36.27	32.85	10.4%	-27.5%	50	793
		No model	35.39	35.39 [~]	32.85	7.7%	-21.9%	50	408
1	e8.5	$\nu_e + \kappa_e$	44.08 [†]	44.09	45.98	4.1%	-23.5%	50	2994
		$\nu_e + \text{const } Pr_t$	43.46 [†]	43.47	45.98	5.5%	-23.2%	50	1395
		No model	49.45	49.85	45.98	8.4%	-21.1%	50	1195
1	e8.75	$\nu_e + \kappa_e$	52.60 [†]	52.52	54.54	3.7%	-20.0%	50	1768
		$\nu_e + \text{const } Pr_t$	52.61 [†]	52.62	54.54	3.5%	-18.8%	50	796
		No model	50.67 [†]	50.70	54.54	7.1%	-19.8%	50	796
10	e7	$\nu_e + \kappa_e$	16.61 [†]	16.64 [~]	16.93	1.9%	-3.4%	2	285
		$\nu_e + \text{const } Pr_t$	17.10 [†]	17.10	16.93	1.0%	-2.4%	2	984
		No model	17.09 [†]	17.09 [~]	16.93	0.9%	-3.6%	2	580

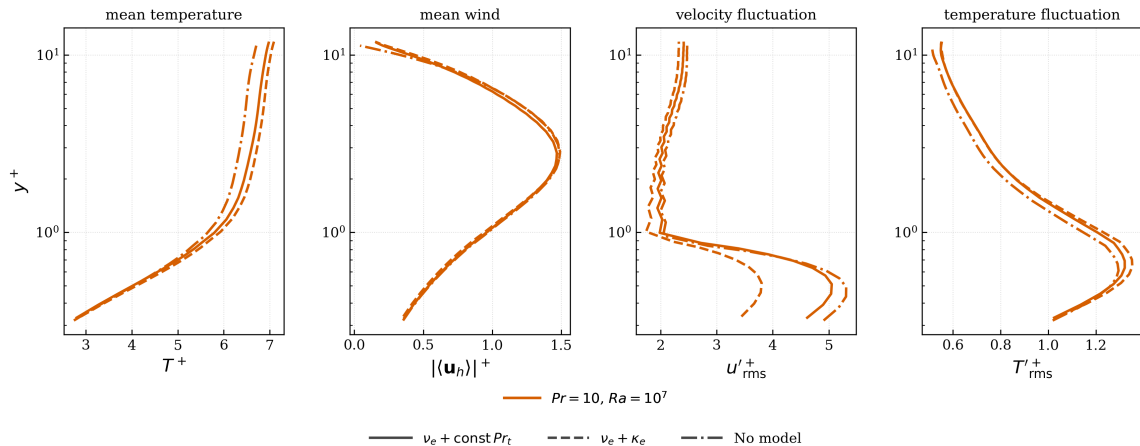
Model comparison



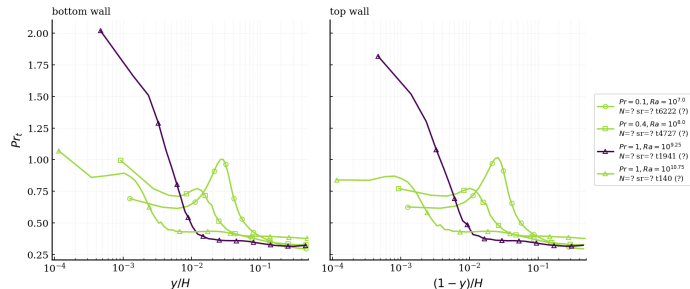
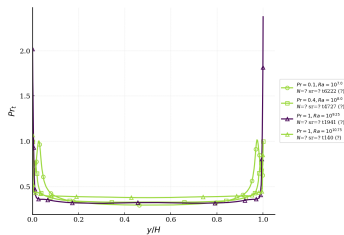
u, u'_{rms} : $\nu_e + Pr_t < \nu_e + \kappa_e < \text{No model}$

T, T'_{rms} : $\text{No model} < \nu_e + \kappa_e < \nu_e + Pr_t$

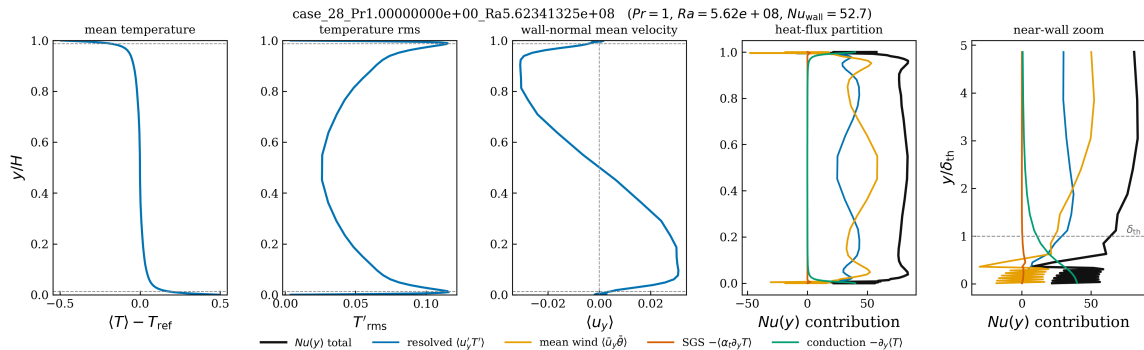
Model comparison



- Minimum-dissipation LES model & symmetry-preserving discretization in turbulent thermal convection
- The numerical scheme accurately predicts heat flux on coarse meshes up to Ra^{15}
- Different LES models give different predictions for the flow and heat quantities.



Zigzag statistically stationary state



Nu_{bot}	Nu_{top}	n	half	Nu_{eT}	Nu_{eTeff}	$eT_{dev}\%$	Nu_{ref}	dev%	$Nu(y)$	std%	$dTH_{dev}\%$	$dU_{dev}\%$	t_{avg}
52.68	52.58	20194	0.29	43.64	36.98	17	54.54	3.5	79.83	3.4	0.0	45.7	$t/t_f = 1728.63$

- Alignment between $\overline{u'T'}$ and $\nabla \bar{T}$
- $Pr_t \stackrel{?}{=} \frac{\varepsilon_u}{\varepsilon_T}$
- Backward cascade
- Forced and mixed convections