A projected Ghost Fluid Method for a mimetic approach to extreme contrast interfaces in multiphase flows

N. Valle[1], F.X. Trias, J. Castro and A. Oliva

Heat and Mass Transfer Technological Centre (CTTC), Universitat Politècnica de Catalunya - BarcelonaTech (UPC)

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Overview

Motivation

Goal

Develop numerical methods for the accurate computation of multiphase flows under extreme contrast interfaces

Application

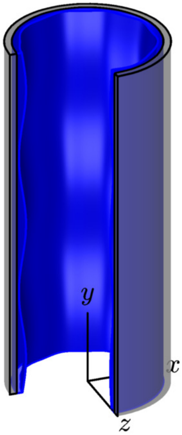
Numerical study of heat and mass transfer in a LiBr falling film absorber for a solar absorption chiller

Challenges

extreme interface density ratio

instabilities at low Reynolds

non-condensable gases reaching the interface



Mono-phase approach

Governing equations

$$\frac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = \nabla \cdot \mathbf{S}$$

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (e \otimes \vec{u}) = \nabla \cdot \lambda \nabla T$$

$$\mathbf{S} = \mu \mathbf{E} - \nabla P$$

$$\mathbf{E} = \frac{\nabla u + \nabla u^T}{2}$$

Multi-phase approach

Jump conditions

$$[\rho]_\Gamma = \rho_l - \rho_v \qquad [\vec{u}]_\Gamma \cdot \hat{n}_\Gamma = -\dot{m}\left[\frac{1}{\rho}\right]_\Gamma$$

$$[\mu]_\Gamma = \mu_l - \mu_v \qquad [\mathbf{S}]_\Gamma \cdot \hat{n}_\Gamma = \sigma\kappa\hat{n}_\Gamma$$

$$[\lambda]_\Gamma = \lambda_l - \lambda_v \qquad [\nabla T]_\Gamma \cdot \hat{n}_\Gamma = L_{vap}\dot{m}$$

Research question

Can we impose the jump conditions above in a ***sharp*** *and* ***physically consistent*** *way?*

Keep all your energy

Symmetry-preserving[1]

$$\begin{aligned}\frac{d}{dt}(u_f^*\Omega_c u_f) = &- u_f^*\cancel{(C(u_f) + C(u_f)^*)}u_f \\ &- u_f^*(D + D^*)u_f \\ &- \cancel{u_f^*(Gp_c) - (Gp_c)^* u_f} = -u_f^*(D + D^*)u_f\end{aligned}$$

$$(\nabla p, \vec{u}) = -(p, \nabla \cdot \vec{u}) \rightarrow (Gp_c)^*\Omega_f u_h = -p_c^*\Omega_c M u_f$$

. . . a symmetry- preserving, spatial discretization of the Navier–Stokes equations is unconditionally stable and conservative.

[1] R.W.C.P. Verstappen and A.E.P. Veldman. "Symmetry-preserving discretization of turbulent flow". In: *J. Comput. Phys.* 187.1 (2003), pp. 343–368.

Metric is all you need

Mimetic method[2]

$$(\vec{u}, \nabla p) = -(\nabla \cdot \vec{u}, p) \quad \rightarrow \quad (u_f, Gp_c) = -(Mu_f, p_c)$$

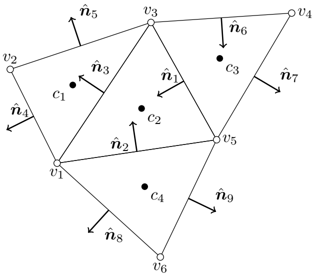
$$(v, w) = \int_\Omega vw \quad \rightarrow \quad (v_h, w_h) = v_h^* \Omega_h w_h$$

$$\int_\Omega \nabla \cdot \vec{u} = \int_{\partial\Omega} \vec{u}\hat{n}_f \approx \sum_f u_f S_f$$

The construction of the derived operators is based on the duality principle . . .

[2] Konstantin Lipnikov, Gianmarco Manzini, and Mikhail Shashkov. "Mimetic finite difference method". In: *J. Comput. Phys.* 257.PB (2014), pp. 1163–1227.

A graph to rule them all



Divergence/primary operator

$$\int_\Omega \nabla \cdot \vec{u} = \int_{\partial\Omega} \vec{u}\hat{n}_f \approx T_{cf} S_f u_f$$

$$T_{cf} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccccccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 \end{array} \\ \begin{pmatrix} 0 & 0 & -1 & +1 & +1 & 0 & 0 & 0 & 0 \\ -1 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 & +1 & +1 \end{pmatrix} \end{array}$$

Summary

Metric

$$\Omega_f = -\Delta_x S_f$$

Symmetry-preserving

$$G = -\Omega_f^{-1} M^* = -(\Delta_x S_f)^{-1} (T_{cf} S_f)^* = -(\Delta_x)^{-1} T_{cf}^*$$

Mimetic

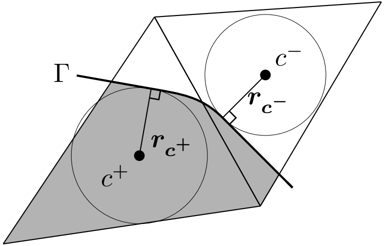
$$(u_h, Gp_c) = -(Mu_h, p_c) = \cancel{u_h^*} \Omega_f G \cancel{p_c} = \cancel{u_h^*} M^* \cancel{p_c}$$

Highlight

The definition of the **metric** of the dual space induces the symmetry-preserving/mimetic **dual operator**

Introduction

Level-Set method



Algorithm[3]

Convolution:

$$\psi_c = \frac{1}{2}\left(tanh\left(\frac{r_c}{2\epsilon}\right) + 1\right)$$

Advection:

$$\frac{\partial \psi_c}{\partial t} + \nabla \cdot (\psi_c \otimes \vec{u}) = 0$$

Recompression:

$$\cancelto{\tau \to \infty}{\frac{\partial \psi_c}{\partial \tau}} + \nabla \cdot (\psi_c(1 - \psi_c)) = \epsilon \nabla^2 \psi_c$$

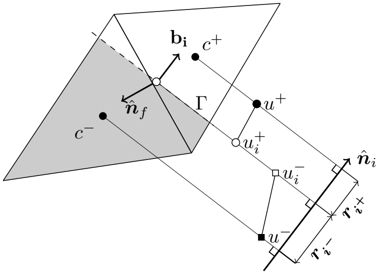
Definition

Level-Set (signed) function:

$$r_c = (x_\Gamma - x_c) \cdot \hat{n}_\Gamma$$

[3] Elin Olsson and Gunilla Kreiss. "A conservative level set method for two phase flow". In: *J. Comput. Phys.* 210.1 (2005), pp. 225–246.

Ghost Fluid Method



Definition

$$\nabla \cdot \lambda \nabla u = f$$
$$[u]_\Gamma = a$$
$$[\lambda \nabla u]_\Gamma \cdot \hat{n}_\Gamma = b$$

Algorithm[4]

$$(\lambda \nabla u)^\pm = (r_{c^\pm}^{-1} T_{pi} - u_\Gamma)^\pm$$
$$[u]_\Gamma = a$$
$$[\lambda \nabla u]_\Gamma \cdot \hat{n}_\Gamma = b$$

Solve for $(\lambda \nabla u)^\pm$

[4] Xu-Dong Liu, Ronald P. Fedkiw, and Myungjoo Kang. "A Boundary Condition Capturing Method for Poisson's Equation on Irregular Domains". In: *J. Comput. Phys.* 160.1 (2000), pp. 151–178.

Ghost Fluid Method

Final forms

$$\nabla \cdot \lambda \nabla u \approx M\hat{\Lambda}_f G u_f - M\hat{\Lambda}_f (\Delta_x)^{-1} Q a_f - \Lambda_f \Omega_c^{-1} H^* cos(\Theta) S_f (\Lambda_f^{\pm})^{-1} b_f$$

$$\lambda \nabla u \approx \hat{\Lambda}_f G u_f - \hat{\Lambda}_f (\Delta_x)^{-1} Q a_f - (\Lambda_f)^{-1} (\Delta_x)^{-1} (R^{\pm})^{-1} \Lambda_f^{\pm} b_f$$

Where:

- Q interface orientation
- $\hat{\Lambda}_f$ harmonic mean of λ

Ghost Fluid Method

Research question

Does this discretization lie in the image of L?

Static phase change

In a static phase change situation, provided Neumann boundary conditions, is the discretization consistent? Condition:

$$\int_\Omega \nabla \cdot \lambda \nabla u = \int_\Omega f = \int_\Gamma [\lambda \nabla u]_\Gamma \cdot \hat{n}_\Gamma \tag{1}$$

However:

$$(\mathbb{1}, f_h + M\hat{\Lambda}_f(\Delta_x)^{-1} Q a_f + \Lambda_f \Omega_c^{-1} H^* cos(\Theta) S_f (\Lambda_f^{\pm})^{-1} b_f) \neq 0$$

Which is not satisfied, unfortunately.

Ghost Fluid Method

Problem

High coefficient ratios magnify this problem!
This may lead to divergence!

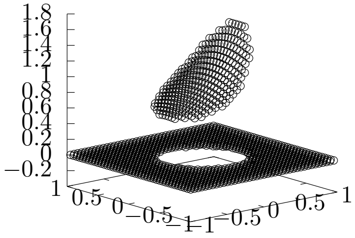


Figure 1: Ghost Fluid Method at $\lambda_+/\lambda_- = 1$.

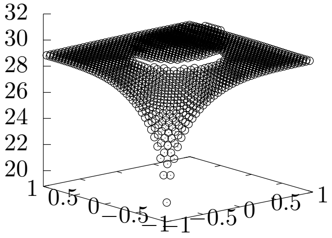


Figure 2: Ghost Fluid Method at $\lambda_+/\lambda_- = 1E4$.

Projected Ghost Fluid Method

Remedy

Force the system $L\vec{u} = \vec{s}$ to have a solution by projecting over the image space.

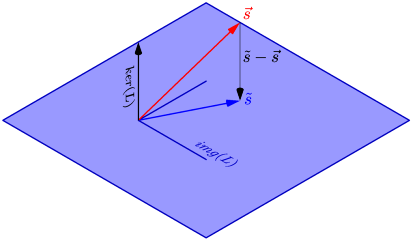


Figure 3: Project $\vec{s}$ over $img(A)$.

Remark

Force the $\vec{s} \in img(L)$ by:

$$\tilde{s} = \vec{s} - \frac{\left\langle \vec{\mathbb{1}}_c, \vec{s} \right\rangle}{|\vec{\mathbb{1}}_c|} \vec{\mathbb{1}}_c \qquad (2)$$

Evenly share the imbalance between all cells.

Projected Ghost Fluid Method

Remedy

Force the system $L\vec{u} = \vec{s}$ to have a solution by projecting over the image space.

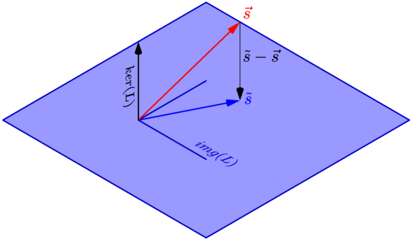


Figure 4: Project $\vec{s}$ over $img(L)$.

Remark

Force the $\vec{s} \in img(L)$ by:

$$\tilde{s} = \vec{s} - \frac{\left\langle \vec{\mathbb{1}}_c, \vec{s} \right\rangle}{|\vec{\mathbb{1}}_a|} \vec{\mathbb{1}}_a \qquad (3)$$

Share the imbalance between adjacent cells.

Projected Ghost Fluid Method

Remedy

Force the system to have a solution by projecting over the image space.

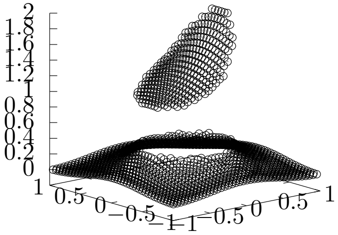


Figure 5: PGFM at $\lambda_+/\lambda_- = 1E4$.

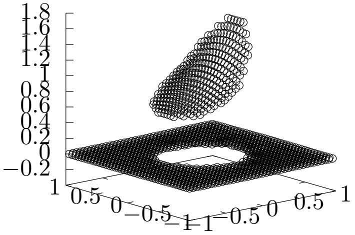


Figure 6: LPGFM at $\lambda_+/\lambda_- = 1E4$.

Projected Ghost Fluid Method

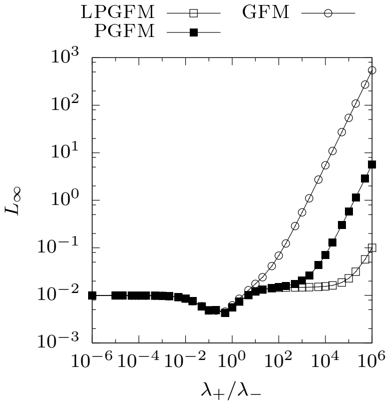


Figure 7: Error norm vs λ_+/λ_-.

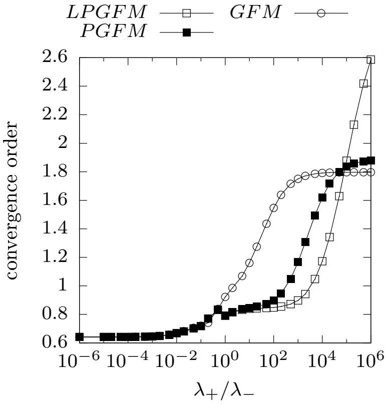


Figure 8: Convergence order vs λ_+/λ_-.

Challenges and opportunities

Pros

- sharp method
- explicit treatment of discontinuities

Cons

- lack of geometric information at the interface
- flow jumps are not in the image of L

Opportunities

- weighted projection
- equivalent interface reconstruction

Thank you for your attention

Ghost Fluid Method

Research question

Does this discretization preserve duality?

Bubble flow

Is the kinetic energy variation due to the pressure jump properly captured by the Ghost Fluid Method?

$$(u_f, Gp_c - (\Delta_x)^{-1} Qa_f) + (Mu_f, p_c) = -u_f^* S_f Qa_f$$

Which is an approximation of the balance between kinetic and surface potential energy.

Ghost Fluid Method

Warning!

Because $(\lambda \nabla u)^{\pm}$ is **defined at the interface**, we need to move it back to the **face** to operate with a regular divergence operator. **This is required to move from a differential approach to an integral one**.

Tangency condition

$$r_{c^+} - r_{c^-} = \hat{n}_\Gamma \cdot (\vec{x}_{c^+} - \vec{x}_{c^-})$$

Homothecy condition

$$\vec{x}_h = \frac{r_{c^+}}{r_{c^+} - r_{c^-}} \vec{x_{c^-}} - \frac{r_{c^-}}{r_{c^+} - r_{c^-}} \vec{x_{c^+}} = H(r_c)\vec{x}_c$$

Ghost Fluid Method

Geometrical Warning!

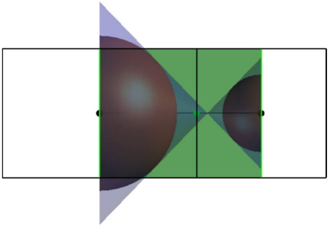
Interface is not uniquely defined!
This prevents us from properly defining the **interface metric**.

Side Note

$$\Omega_i = S_f |\vec{x_h} - \vec{x_{c^+}}|$$

If the following holds: [5]

- cell and face centroids are aligned
- plane does not intersect the base



[5] Murray S. Klamkin. "On the volume of a class of truncated prisms and some related centroid problems". In: *Math. Assoc. Am.* 41.4 (1968), pp. 175–181.