Symmetry-preserving discretizations in unstructured staggered meshes

N. Valle[1,2], F.X. Trias[1], and R.W.C.P Verstappen[2]

[1]Heat and Mass Transfer Technological Center, Technical University of Catalonia, ETSEIAT, c/Colom 11, 08222 Terrassa, Spain
[2]Bernoulli Institute, University of Groningen, PO Box 407, 9700AK Groningen, the Netherlands

April 6, 2021

Abstract

The adoption of symmetry-preserving discretizations is presented in terms of the collocated, unstructured meshes customary of commercial codes. By adopting an algebraic approach, a discretization of the convective terms that reduces to the well know staggered method of Harlow and Welch is presented. The scheme properties are presented along with benchmark simulations concerning turbulent flows, achieving exact conservation of momentum and kinetic energy.

Since the pioneering work of Harlow and Welch [1], the use of staggered variables has gained widespread acceptance within the scientific community due to its superior properties for the simulation of incompressible flows. Its use along with symmetry-prserving schemes [2, 3] sets the standard of high quality, state-of-the-art Direct Numerical Simulation of turbulent flows. However, despite its known advantages, the popularity drops dramatically in commercial, unstructured codes. The main reason behind it is the complex formulation required to move from a structured arrangement to an unstructured one when it comes to construct an overlapping, staggered mesh.

While several attempts have been made in the past to bring these ideas into unstructured meshes, there is none, to the best of our knowledge, that recovers the original Harlow and Welch formulation when applied to a structured one. Perot [4] and Zhang et al. [5] discussed its implementation by interpolating the collocated discretization to the faces, while numerical patologies appeared on its way. Later on, Hicken et al. [6] introduced a new type of shift transformations to address such a problem. However, none of the methods introduced above recovers the original Harlow and Welch formulation when applied to structured meshes. In addition, the use of back and forth interpolation modifies the spectra of the operator such that it compromises its reliability and efficiency, as it can be seen in Figure 1.

An alternative approach involves the construction of collocated discretizations that preserve symmetry [7]. Nonetheless, this clashes with the solution of the compact Laplace equation, which solves pressure by enforcing null divergence of the staggered velocity [8]. While this mismatch can be fixed by interpolation, this may enlarge the kernel dimension of the overall linear system of equations, giving birth to the well-known checker board problem. While numerical remedies can be found, such as the popular Rhie and Chow method [9], this introduces a mass conservation error [8].

A remarkable attempt to bring staggered formulations into unstructured meshes was the adoption of the vorticity formulation. By locating vorticity at the mesh edges it was better suited for an unstructured mesh. Nonetheless, as was shown by Horiuti and Itami [10], this formulation does not collapse to the

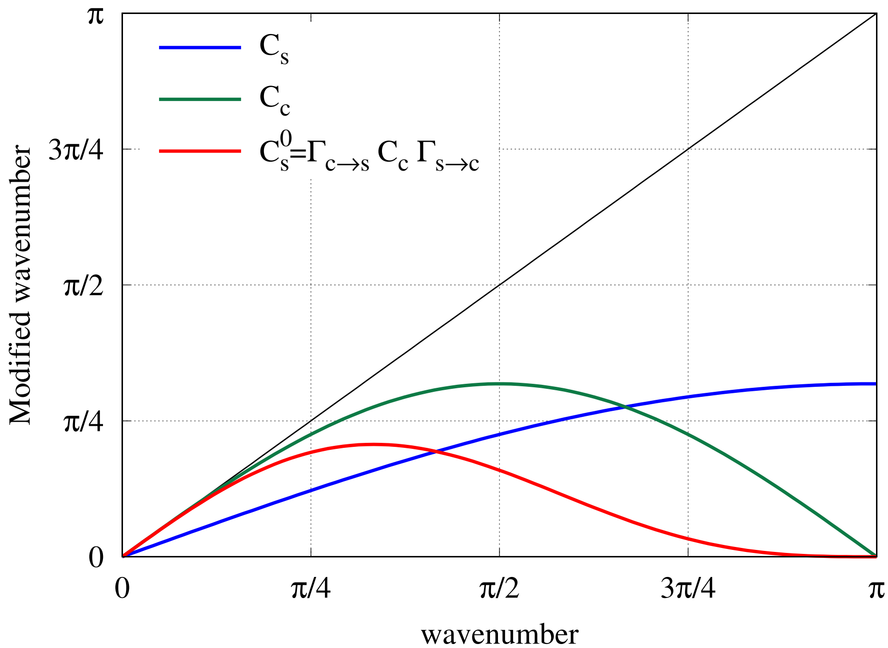

Figure 1: Dispersion errors for staggered, collocated and interpolated convective operators.

skew-symmetric one unless homogeneous Cartesian meshes are used. In addition, as presented by Zhang et al. [5], this formulation does indeed preserve kinetic energy and vorticity, while it is unsuitable for the conservation of kinetic energy and linear momentum at the same time, suggesting the use of the divergence form instead.

However, the geometric intuition behind the rotational form was indeed to circumvent the construction of an explicit staggered mesh, which may turn cumbersome in unstructured meshes. Because the ultimate reason between the mismatch between rotational and discrete forms at the discrete level was the lack of a discrete chain rule [10]. In this work, we embrace the idea of computing quantities at every face edge but enforce the fulfillment of the discrete conservation form. By doing so, we present a new discretization of the convective term which recovers the classical Harlow and Welch formulation when applied to a Cartesian mesh, but that is also suitable for unstructured meshes.

Equipped with such a discretization, we tackle the simulation of canonical problems, such as the well-known turbulent channel flow problem, showing the expected converge for second order, staggered discretizations.

References

[1] Francis H. Harlow and J. Eddie Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *Physics of Fluids*, 8(12):2182–2189, 1965.

[2] R. W. C. P. Verstappen and A. E. P. Veldman. Direct numerical simulation of turbulence at lower costs. *Journal of engineering mathematics*, (c):143–159, 1997.

[3] R. W. C. P. Verstappen and A. E. P. Veldman. Symmetry-preserving discretization of turbulent flow. *Journal of Computational Physics*, 187(1):343–368, 2003.

[4] Blair Perot. Conservation Properties of Unstructured Staggered Mesh Schemes. *Journal of Computational Physics*, 159(1):58–89, 2000.

[5] Xing Zhang, David Schmidt, and Blair Perot. Accuracy and conservation properties of a three-dimensional unstructured staggered mesh scheme for fluid dynamics. *Journal of Computational Physics*, 175(2):764–791, 2002.

[6] J. E. Hicken, F. E. Ham, J. Militzer, and M. Koksal. A shift transformation for fully conservative methods: Turbulence simulation on complex, unstructured grids. *Journal of Computational Physics*, 208(2):704–734, 2005.

[7] F. X. Trias, O. Lehmkuhl, A. Oliva, C. D. Pérez-Segarra, and R. W. C. P. Verstappen. Symmetry-preserving discretization of Navier–Stokes equations on collocated unstructured grids. *Journal of Computational Physics*, 258:246–267, feb 2014.

[8] F. Trias, N. Valle, A. Gorobets, and A. Oliva. Symmetry-Preserving Discretization of Navier-Stokes On Unstructured Grids: Collocated Vs Staggered. (January):11–15, 2021.

[9] C. M. Rhie and W. L. Chow. Numerical study of the turbulent flow past an airfoil with trailing edge separation. *AIAA Journal*, 21(11):1525–1532, 1983.

[10] Kiyosi Horiuti and Takao Itami. Truncation Error Analysis of the Rotational Form for the Convective Terms in the Navier-Stokes Equation. *Journal of Computational Physics*, 145(2):671–692, 1998.