On a Conservative Solution to Checkerboarding: Examining the Causes of Non-physical Pressure Modes

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Contents

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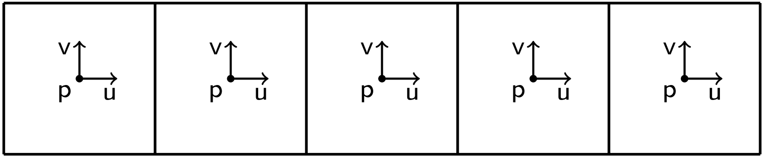
Decoupling of pressure

- Collocated grid arrangement
- Central differencing discretisation
- Wide-stencil gradient, divergence & Laplacian



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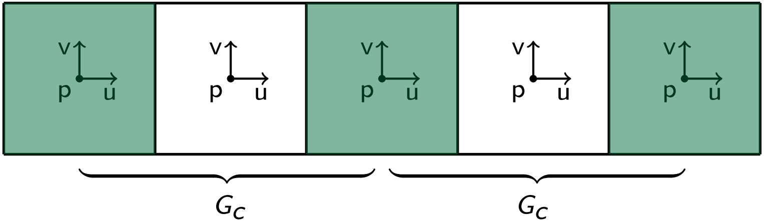

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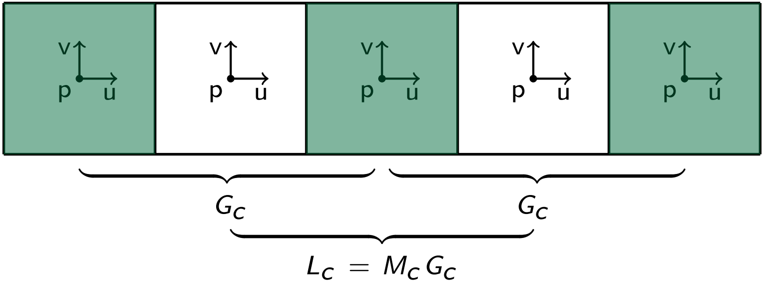

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Table: Occurence of and solutions to checkerboarding in the fractional step method

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Pressure error

Pressure error

- The pressure error is linked to the divergence of $\mathbf{u}$
- For the compact stencil method: $M\Gamma_{cs}\mathbf{u}_c^{n+1} \neq \mathbf{0}_c$
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$$M\mathbf{u}_s^{n+1} = \mathbf{0}_c$$

$$\begin{aligned}
M_c\mathbf{u}_c^{n+1} &= M\Gamma_{cs}(\mathbf{u}_c^p - G_c\tilde{\mathbf{p}}_c') \\
&= M(\Gamma_{cs}\mathbf{u}_c^p - G\tilde{\mathbf{p}}_c') + M(G\tilde{\mathbf{p}}_c' - \Gamma_{cs}G_c\tilde{\mathbf{p}}_c') \\
&= (L - L_c)\tilde{\mathbf{p}}_c'
\end{aligned}$$

Objective

Questions

- What are the origins of the oscillations?
- How can we quantify the oscillations?
- Can we design a method that diminishes checkerboarding with less numerical dissipation?

Mechanism 1: $\Delta t \to 0^+$

As an example, consider:

- Compact-stencil Laplacian method
- No pressure predictor, i.e. $\tilde{\mathbf{p}}_c^p = \mathbf{0}_c$

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&= \mathbf{u}_c^0 - G_c \sum_{i=1}^{n+1} \tilde{\mathbf{p}}_c^i \\
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L\tilde{\mathbf{p}}_c^{n+2} &= M_c \mathbf{u}_c^0 - L_c \mathbb{P}_c^{n+1} \\
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Stationary iterative solver

$L_c \mathbb{P}_c = M_c \mathbf{u}_c^0$

$\to$ Allows for checkerboarding!

Mechanism 2: $\mathbf{p}_c^p = \mathbf{p}_c^n$

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Similar problem

- Solving a wide-stencil Laplacian if $\tilde{\mathbf{p}}_c' \to \mathbf{0}_c$
- In case of a steady-state solution
- In case $\Delta t \to 0^+$, combines well with mechanism 1

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Choice in Poisson solver

- Approximate inverse of L can produce oscillations if $Im(\tilde{L}^{-1}) \not\perp Ker(L_c)$

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Concluding mechanism

- $\tilde{L}^{-1}$, or rather $\overline{L}^{-1}$ can produce oscillations
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Similarly, preconditioners can cause oscillations

- $Q_L^{-1} L Q_R^{-1} \tilde{\mathbf{q}}_c^{n+1} = Q_L^{-1} M_c \mathbf{u}_c^p$
- where $Q_R^{-1} \tilde{\mathbf{q}}_c^{n+1} = \tilde{\mathbf{p}}_c^{n+1}$
- if $Im(Q_R^{-1}) \not\perp Ker(L_c)$

Mechanism 4: Non-symmetries of operators

Inconsistent operators

- Symmetry-preserving: $M_c = -(\Omega G_c)^T$
- If not, $L_c = M_c G_c \neq L_c^T$
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Oscillations can then enter via the right hand side of the Poisson equation:

$$\begin{aligned} \mathbf{u}_c^p &= \mathbf{u}_c^n - \Delta t[Con + Dif] - G_c \tilde{\mathbf{p}}_c^p \\ \mathbf{u}_c^p &= \mathbf{u}_c^{p,n-1} - G_c \tilde{\mathbf{p}}_c^n - \Delta t[Con + Dif] - G_c \tilde{\mathbf{p}}_c^p \\ M_c \mathbf{u}_c^p &= M_c(\mathbf{u}_c^{p,n-1} - \Delta t[Con + Dif]) - L_c(\tilde{\mathbf{p}}_c^n + \tilde{\mathbf{p}}_c^p) \end{aligned} \qquad M_c G_c = L_c$$

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$$

This also means SP-method automatically filters these type of oscillations!

Mechanisms of interest

Focus on mechanism 1 & 2

- ✓ 1. $\Delta t \to 0^+$
- ✓ 2. $\tilde{\mathbf{p}}_c^p = \theta_p \tilde{\mathbf{p}}_c^n, \quad \theta_p \in [0, 1]$
- × 3. Poisson solver
- × 4. Non-symmetries of operators

$\left.\begin{array}{l}\text{3. Poisson solver}\\ \text{4. Non-symmetries of operators}\end{array}\right\}$ limited oscillations observed

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Related to $Ker(L_c)$
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- Oscillations occur locally $\rightarrow$ (nearly/fully) orthogonal to $Ker(L_c)$

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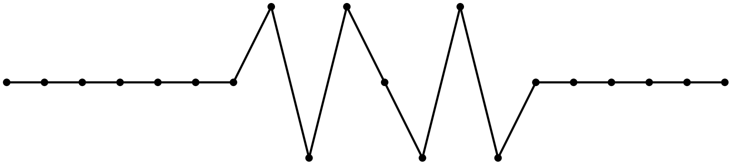

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Starting from the pressure budget term:

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Which is strictly dissipative. Dividing out $\Delta t \mathbf{p}_c^T L \mathbf{p}_c$:

$$C_{cb} = 1 - \frac{\mathbf{p}_c^T L_c \mathbf{p}_c}{\mathbf{p}_c^T L \mathbf{p}_c}$$

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Broader definition

Starting from the pressure budget term:

$$-\mathbf{u}_c^T \Omega G_c \mathbf{p}_c = \mathbf{p}_c^T M_c \mathbf{u}_c = \Delta t \mathbf{p}_c^T (L - L_c) \mathbf{p}_c \in [\Delta t \mathbf{p}_c^T L \mathbf{p}_c, 0]$$

Which is strictly dissipative. Dividing out $\Delta t \mathbf{p}_c^T L \mathbf{p}_c$:

$$C_{cb} = 1 - \frac{\mathbf{p}_c^T L_c \mathbf{p}_c}{\mathbf{p}_c^T L \mathbf{p}_c} = 1 - \frac{\mathbf{p}_c^T G_c^T \Omega G_c \mathbf{p}_c}{\mathbf{p}_c^T G^T \Omega_s G \mathbf{p}_c} = 1 - \frac{||G_c \mathbf{p}_c||}{||G \mathbf{p}_c||} \in [0, 1] \begin{cases} 0, & \text{smooth} \\ 1, & \text{fully in } Ker(L_c) \end{cases}$$

Checkerboard coefficient C_{cb}

- Global, non-dimensional, normalised, time-step independent
- Able to detect local oscillations

θ_{cb}-solver

We set θ_p in the momentum predictor:

$$\mathbf{u}_c^p = R(\mathbf{u}_c^n, \mathbf{u}_s^n) - G_c \tilde{\mathbf{p}}_c^p = R(\mathbf{u}_c^n, \mathbf{u}_s^n) - G_c \theta_p \tilde{\mathbf{p}}_c^n$$

Which is also non-dimensional and $\in [0, 1]$. A new solver can be derived by setting θ_p dynamically as $\theta_p = 1 - C_{cb}$.

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- Higher θ_p is a known cause of checkerboarding
- Negative feedback through C_{cb} can auto-regulate the problem

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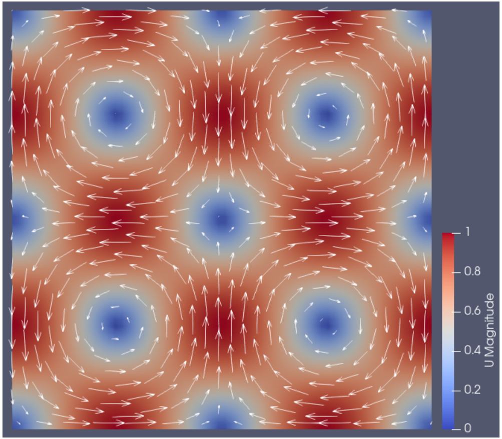
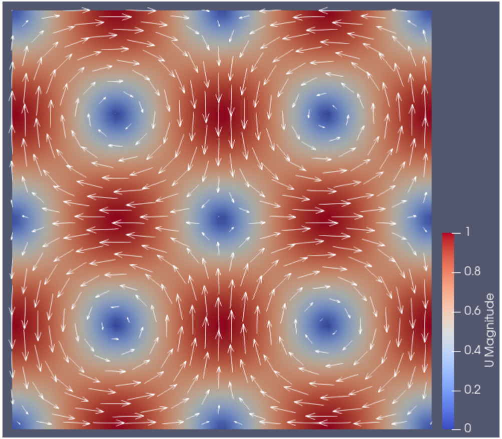
Overview of tested solvers:

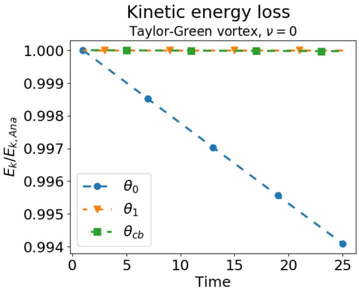
Solver	θ_0	θ_1	θ_{cb}
θ_p	0	1	$1 - C_{cb}$

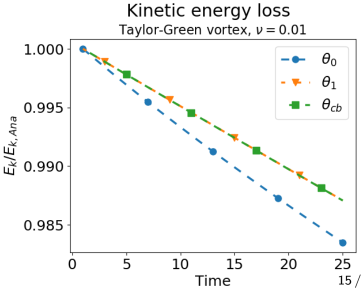
2D Taylor-Green vortex



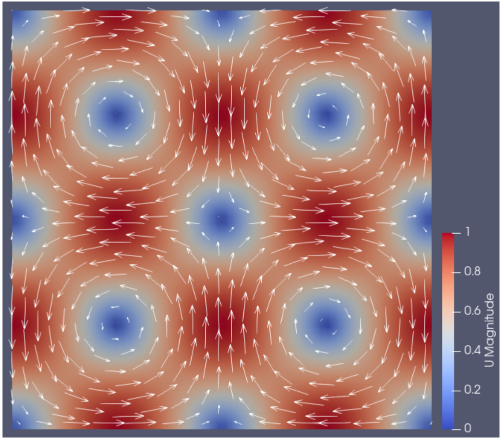
2D Taylor-Green vortex

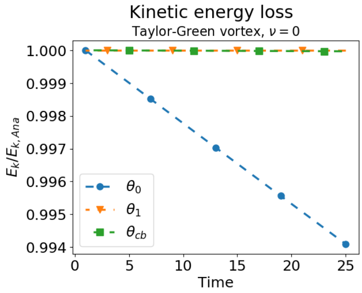


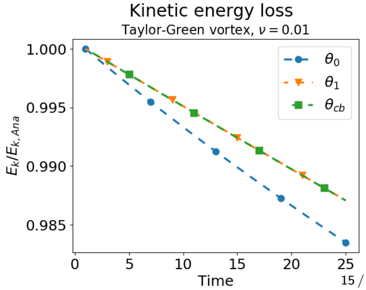




2D Taylor-Green vortex



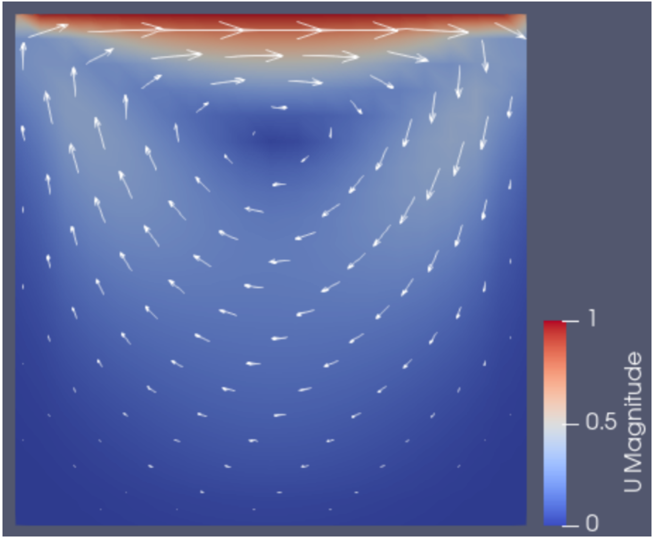




Conclusions

- At no oscillations $\theta_{cb} \rightarrow \theta_1$
- θ_{cb} and θ_1 almost completely free from numerical dissipation

Lid-driven cavity



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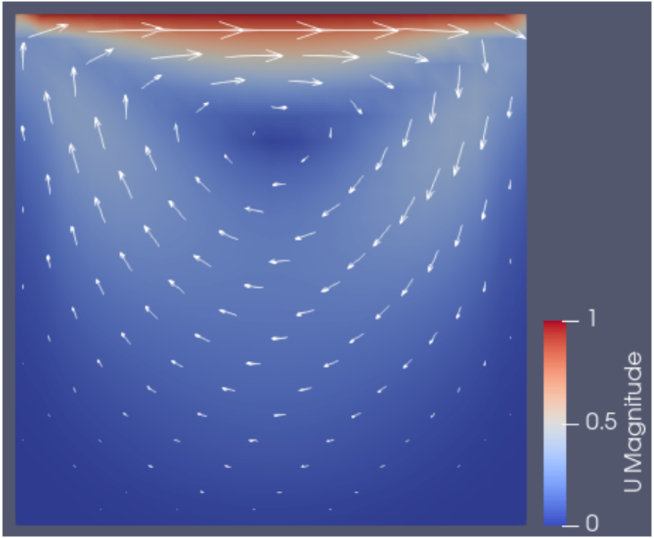

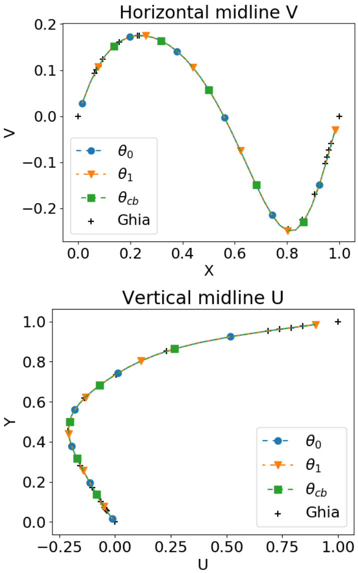

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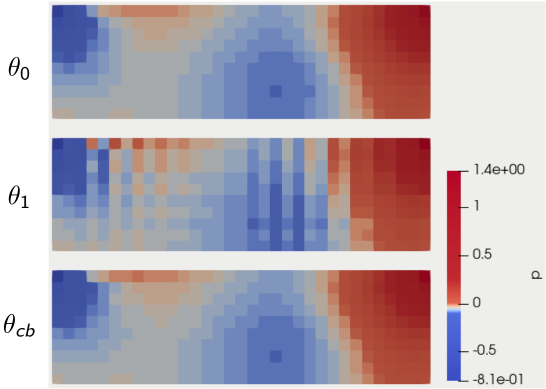

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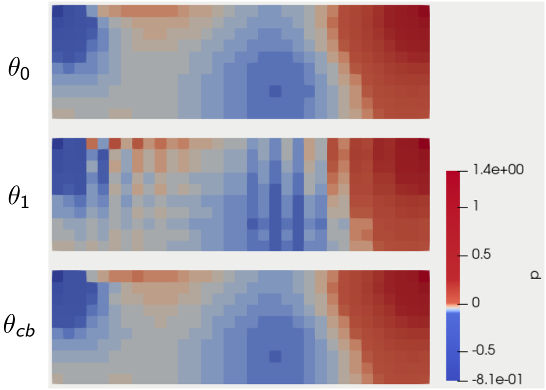

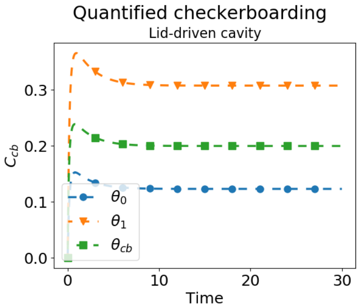

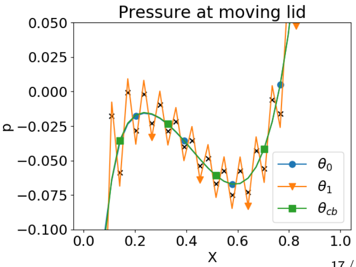

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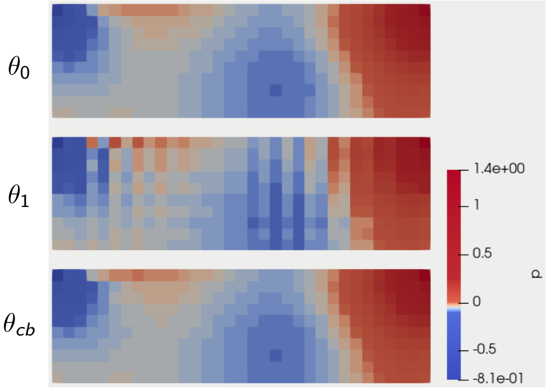
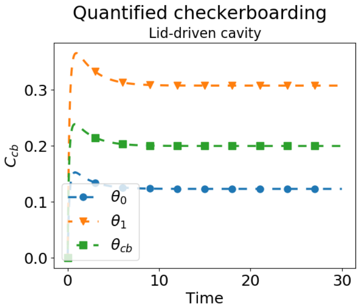
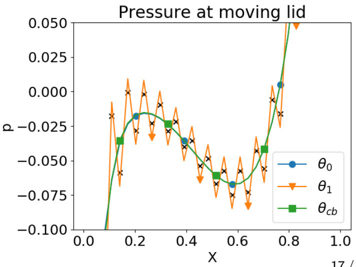


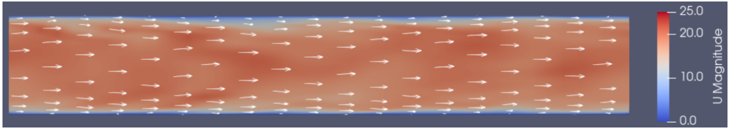




Conclusions

- θ_1 shows more oscillations, portrayed by C_{cb}
- θ_{cb} adequately lowers through feedback
- $Ker(L_c)$ filtering is inadequate

Channel flow at $Re_\tau = 180$



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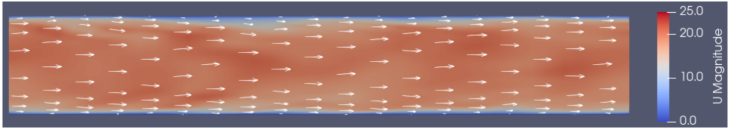

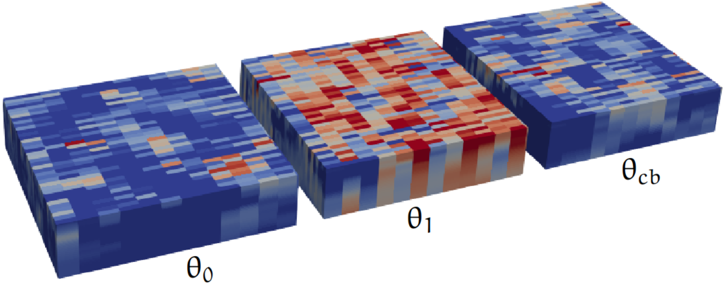

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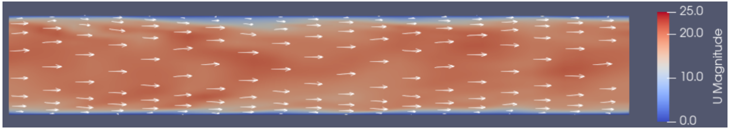

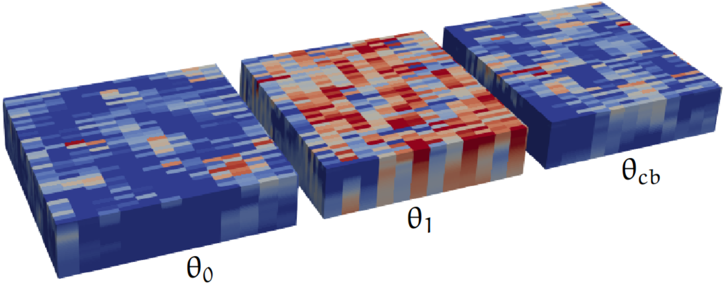

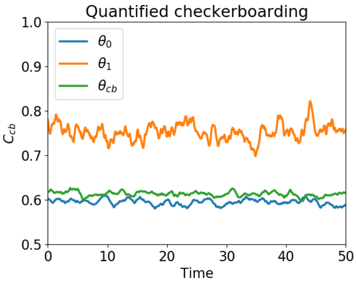

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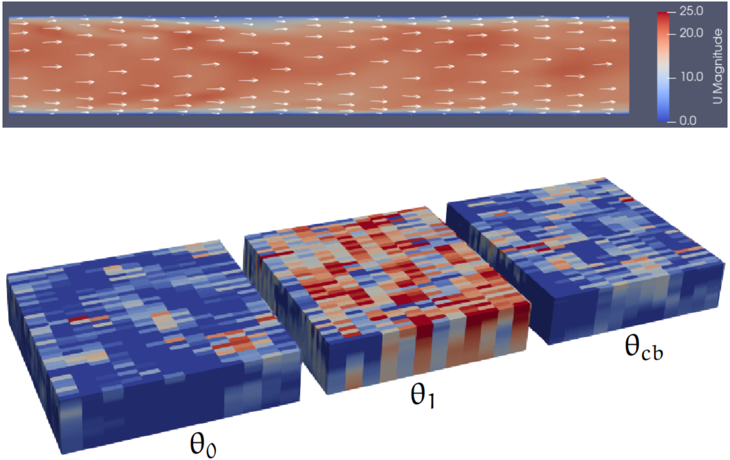

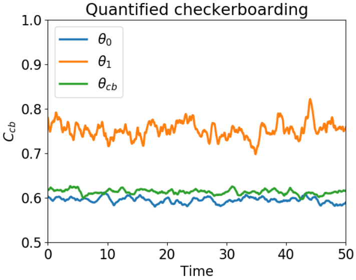

Conclusions

- More oscillations in general, most for θ_1
- θ_{cb} settles closer to θ_0

Conclusions

Origins of checkerboarding

- $\Delta t \to 0^+$
- $\tilde{\mathbf{p}}_c^p = \theta_p \tilde{\mathbf{p}}_c^n, \quad \theta_p \in [0, 1]$
- Poisson solver
- Non-symmetries of operators

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- $Ker(L_c)$ is inadequate for quantifying and filtering
- C_{cb} offers a global non-dimensional normalised coefficient, independent of time-step

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Outlook

- Can we use a local C_{cb} to diminish oscillations locally?
- How does it compare to other, generalised Rhie-Chow interpolation methods?
- What if the origin is different form $\tilde{\mathbf{p}}_c^p$?

Questions?

Thank you for attending!

Any questions?