Energy-consistent discretization of viscous dissipation with application to natural convection flows

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Abstract. An energy-consistent formulation of incompressible natural convection flows is proposed. In particular, a discretization of the viscous dissipation term is derived which is consistent with the local kinetic energy equation that follows from the momentum equation. Simulations of the Rayleigh-Taylor problem show that total energy is indeed conserved with the new formulation, in contrast to the case where viscous dissipation is neglected.

1 Introduction

The presence of diffusion in the Navier-Stokes equations causes dissipation, acting as a sink of kinetic energy and a source of internal energy. Evaluating the dissipation function locally is for example important in studying natural convection flows with large length scales, such as in the Earth mantle [1, 2]. In this paper, we include the viscous dissipation term in the internal energy equation such that we get a correct global energy balance. We propose a discrete dissipation operator, and show that it cannot be chosen freely but is *implied* by the discretization of the viscous terms in the momentum equations and by the definition of the kinetic energy. This discrete dissipation operator is not only of use in the internal energy equation, but also useful beyond the context of natural convection flows, e.g. when estimating the dissipation of kinetic energy in turbulent flows in a numerical simulation. We also propose a time integration method that preserves the total energy balance upon time marching.

2 Methodology

We are studying incompressible flow under the Boussinesq approximation, consisting of the mass, momentum and internal energy equations:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\rho_0 \left(\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) \right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} - \rho_0 \beta (T - T_0) \boldsymbol{g}, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho_0 c T) + \nabla \cdot (\boldsymbol{u}(\rho_0 c T)) = \mu \Phi + \lambda \nabla^2 T. \tag{3}$$

The focus here is on the viscous dissipation term $\Phi := \|\nabla \boldsymbol{u}\|^2$, which is commonly ignored in equation (3). The viscous dissipation is responsible for a decrease in kinetic energy and an increase in internal energy, which for a closed box with no-slip boundary conditions reads:

$$\frac{\mathrm{d}E_k}{\mathrm{d}t} = -\mu \int_\Omega \Phi \mathrm{d}\Omega + \int_\Omega \beta g \rho_0 (T - T_0) v \mathrm{d}\Omega, \tag{4}$$

$$\frac{\mathrm{d}E_i}{\mathrm{d}t} = \mu \int_\Omega \Phi \mathrm{d}\Omega + \int_{\partial\Omega} \lambda \nabla T \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{5}$$

such that the sum of the two cancels when considering the global energy equation:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E_k}{\mathrm{d}t} + \frac{\mathrm{d}E_i}{\mathrm{d}t} = \int_\Omega \beta g \rho_0 (T - T_0) v \mathrm{d}\Omega + \int_{\partial\Omega} \lambda \nabla T \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{6}$$

In order to have a discrete equivalent of (4)-(6), we need a carefully designed discretization method, which will be obtained by extending the well-known energy-conserving ('MAC') discretization on a staggered grid [3]. The staggered grid discretization naturally leads to the following *local* kinetic energy definition:

$$k_{i,j} := \frac{1}{4} u^2_{i+1/2,j} + \frac{1}{4} u^2_{i-1/2,j} + \frac{1}{4} v^2_{i,j+1/2} + \frac{1}{4} v^2_{i,j-1/2}. \tag{7}$$

This kinetic energy definition, together with the momentum equations, implies the following discrete dissipation function

$$\Phi_{i,j} = \frac{1}{4} \Phi^u_{i+1/2,j} + \frac{1}{4} \Phi^u_{i-1/2,j} + \frac{1}{4} \Phi^v_{i,j+1/2} + \frac{1}{4} \Phi^v_{i,j-1/2}, \tag{8}$$

where

$$\begin{aligned} \Phi^u_{i+1/2,j} = &- \left(\frac{u_{i+3/2,j} - u_{i+1/2,j}}{\Delta x} \right)^2 - \left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} \right)^2 \\ &- \left(\frac{u_{i+1/2,j+1} - u_{i+1/2,j}}{\Delta y} \right)^2 - \left(\frac{u_{i+1/2,j} - u_{i+1/2,j-1}}{\Delta y} \right)^2 \end{aligned} \tag{9}$$

$$\begin{aligned} \Phi^v_{i,j+1/2} = &- \left(\frac{v_{i+1,j+1/2} - v_{i,j+1/2}}{\Delta x} \right)^2 - \left(\frac{v_{i+1,j-1/2} - v_{i,j-1/2}}{\Delta x} \right)^2 \\ &- \left(\frac{v_{i,j+3/2} - v_{i,j+1/2}}{\Delta y} \right)^2 - \left(\frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y} \right)^2 \end{aligned} \tag{10}$$

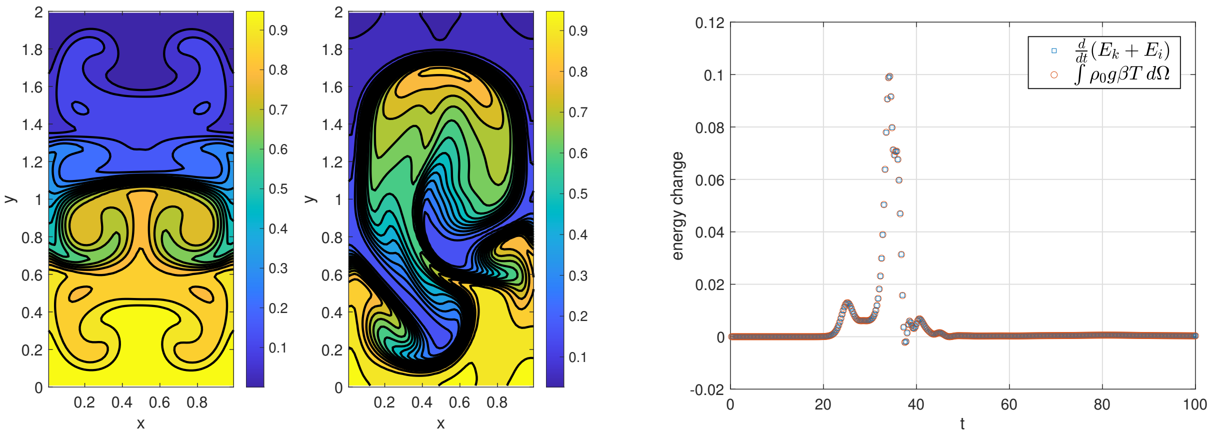
We employ the implicit midpoint method to integrate the spatially discretized equations in time. Based on (i) our proposed discrete dissipation function (8), (ii) the energy-conserving nature of the implicit midpoint method [4], (iii) the compatibility of the divergence and gradient operators on a staggered grid, and (iv) the skew-symmetry of the convective terms, we obtain a discrete energy balance that mimics (6):

$$\frac{E_h^{n+1} - E_h^n}{\Delta t} = \beta g \rho_0 (V_h^{n+1/2})^T (A T_h^{n+1/2} + y_T) + \lambda 1^T (D_T T_h^{n+1/2} + \hat{y}_T). \tag{11}$$

Here, $E_h = E_{k,h} + E_{i,h} = \frac{1}{2} V_h^T \Omega_V V_h + 1^T \Omega_p T_h$ is the discrete approximation to the total energy, A is an averaging operator from temperature locations to velocity locations, D_T is the discrete diffusion operator for the temperature, and y_T and $\hat{y}_T$ take boundary conditions into account.

3 Results

We simulate the well-known Rayleigh-Taylor problem, which features a square box (all boundaries no-slip and adiabatic) with a cold fluid on top of a warm fluid. The domain size is 1×2, the grid is 64×128, the time step $\Delta t = 5 \cdot 10^{-3}$ and the end time $T = 100$. There are 3 dimensionless quantities: the Rayleigh number $\mathrm{Ra} = 10^6$, the Prandtl number $\mathrm{Pr} = 0.71$ and the Gebhart number $\mathrm{Ge} = 10^{-1}$. The Gebhart number is defined as $\mathrm{Ge} = \frac{\beta g H}{c}$ and arises due to inclusion of viscous dissipation [1]. After an initial instability has developed, an asymmetry in the solution appears around $t = 30$, triggering a sequence of well-known 'mushroom' type plumes: hot plumes rising upward and cold plumes sinking downward (figures 1a-1b). The viscous dissipation causes an increase in average temperature. Most existing natural convection models, which ignore the viscous dissipation term, would not predict such a temperature increase. In our model the temperature increase *exactly* matches the kinetic energy loss through viscous dissipation, and this is confirmed by figure 1c, which shows the different terms in equation (11).



(a) $t = 30$, $\mathrm{Ge} = 0.1$

(b) $t = 35$, $\mathrm{Ge} = 0.1$

(c) Energy exchange: kinetic+internal versus potential energy.

Figure 1: Rayleigh-Taylor simulation results.

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