Energy consistent discretization of viscous dissipation with application to natural convection

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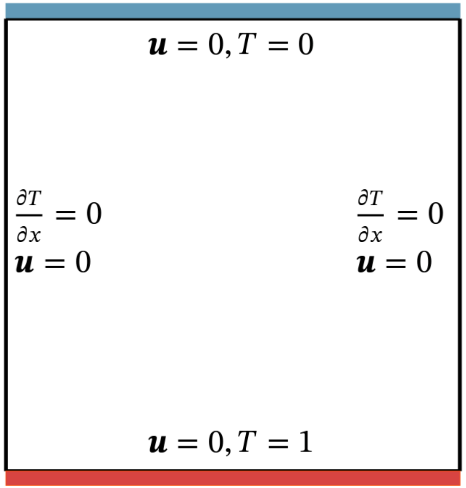
CWI

Rayleigh-Bénard convection

- “granddaddy of canonical examples used to study pattern formation and behavior in spatially extended systems” [1]
- “a hallmark flow beloved by fluid dynamicists and mathematicians alike for its analytical tractability, yet rich behaviour” [2]
- Relevance for many geophysical and astrophysical flows [3]:
  - Convection in atmospheric boundary layer
  - Convection in Earth mantle



[1] A.C. Newell, T. Passot, and J. Lega, Order parameter equations for patterns, Ann. Rev. Fluid Mech., 25: 399 - 453, 1993.
[2] https://blogs.egu.eu/divisions/as/2019/09/20/a-simple-model-of-convection-to-study-the-atmospheric-surface-layer/
[3] A. Pandey, J. Scheel, and J. Schumacher. Turbulent superstructures in Rayleigh-Bénard convection. Nature Communications, 9:2118, 2018.

Governing equations: mass + momentum

- Mass + momentum conservation, “incompressible”

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\rho_0 \left(\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) \right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{g}$$

- **Boussinesq: density variations only important in gravity term**
- **Density varies only with temperature**, not with pressure

$$\rho(T) = \rho_0 - \beta \rho_0 (T - T_0)$$

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p$$

Governing equations: internal energy

- Temperature follows from internal energy equation:

$$\frac{\partial}{\partial t}\underbrace{(\rho_0 c T)}_{e_i} + \nabla \cdot (\boldsymbol{u}(\rho_0 c T)) = \mu \|\nabla \boldsymbol{u}\|^2 + \lambda \nabla^2 T,$$

viscous dissipation $\Phi = \|\nabla \boldsymbol{u}\|^2$

Note:

- If density = constant, then mass + momentum decouple from internal energy equation. Our approach still applicable for that case.
- Three equations, three unknowns ($\boldsymbol{u}$, p, T)

Kinetic, internal, total energy

- Kinetic energy: momentum x ***u***

$$\frac{\partial}{\partial t}\underbrace{(\frac{1}{2}\rho_0|\boldsymbol{u}|^2)}_{e_k} + \nabla\cdot(\frac{1}{2}\rho_0|\boldsymbol{u}|^2\boldsymbol{u}) = -\boldsymbol{u}\cdot\nabla p + \mu\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) - \mu\|\nabla\boldsymbol{u}\|^2 + \rho\boldsymbol{g}\cdot\boldsymbol{u}$$

- Internal energy

$$\frac{\partial}{\partial t}\underbrace{(\rho_0 cT)}_{e_i} + \nabla\cdot(\boldsymbol{u}(\rho_0 cT)) = \mu\|\nabla\boldsymbol{u}\|^2 + \lambda\nabla^2 T,$$

- Total energy (kinetic + internal)

$$\frac{\partial}{\partial t}\underbrace{(\frac{1}{2}\rho_0|\boldsymbol{u}|^2 + \rho_0 cT)}_{e=e_k+e_i} + \nabla\cdot((\frac{1}{2}\rho_0|\boldsymbol{u}|^2 + \rho_0 cT)\boldsymbol{u}) = -\nabla\cdot(p\boldsymbol{u}) + \mu\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \rho\boldsymbol{g}\cdot\boldsymbol{u} + \lambda\nabla^2 T$$

Kinetic, internal, total energy

- Common assumption: neglect $\Phi = \|\nabla \boldsymbol{u}\|^2$

$$\frac{\partial}{\partial t}(\underbrace{\frac{1}{2}\rho_0|\boldsymbol{u}|^2}_{e_k}) + \nabla\cdot(\frac{1}{2}\rho_0|\boldsymbol{u}|^2\boldsymbol{u}) = -\boldsymbol{u}\cdot\nabla p + \mu\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) - \mu\|\nabla\boldsymbol{u}\|^2 + \rho\boldsymbol{g}\cdot\boldsymbol{u}$$

$$\frac{\partial}{\partial t}(\underbrace{\rho_0 cT}_{e_i}) + \nabla\cdot(\boldsymbol{u}(\rho_0 cT)) = \lambda\nabla^2 T,$$

$$\frac{\partial}{\partial t}(\underbrace{\frac{1}{2}\rho_0|\boldsymbol{u}|^2 + \rho_0 cT}_{e=e_k+e_i}) + \nabla\cdot((\frac{1}{2}\rho_0|\boldsymbol{u}|^2 + \rho_0 cT)\boldsymbol{u}) = -\nabla\cdot(p\boldsymbol{u}) + \mu\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \rho\boldsymbol{g}\cdot\boldsymbol{u} + \lambda\nabla^2 T - \mu\|\nabla\boldsymbol{u}\|^2$$

- Total energy conservation is violated

$$\kappa = \frac{\lambda}{\rho_0 c}$$

Importance of Φ: non-dimensionalization

- Non-dimensionalize:

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{H}, \qquad \tilde{t} = \frac{t u_{\mathrm{ref}}}{H}, \qquad \tilde{\boldsymbol{u}} = \frac{\boldsymbol{u}}{u_{\mathrm{ref}}}, \qquad \tilde{T} = \frac{T - T_0}{\Delta T}, \qquad \tilde{p}' = \frac{p'}{\rho_0 u_{\mathrm{ref}}^2}$$

- With Φ included, we have 3 dimensionless quantities (instead of 2)

$$\mathrm{Ra} = \frac{\beta g \Delta T H^3}{\nu\kappa} \qquad \mathrm{Pr} = \frac{\nu}{\kappa} \qquad \mathrm{Ge} = \frac{\beta g H}{c}$$

- For air at atmospheric conditions:

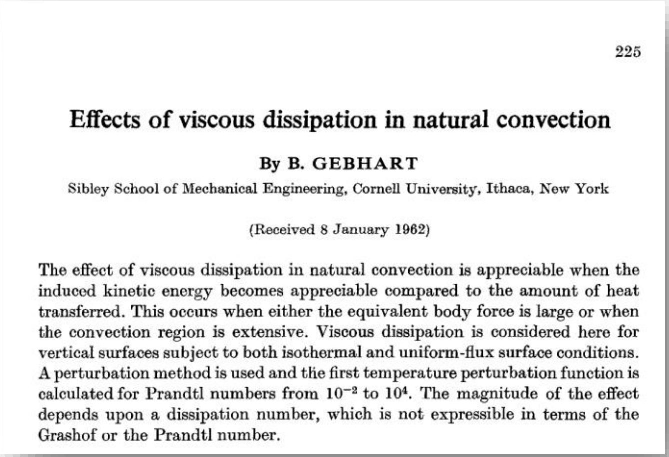
$$\beta \approx 10^{-3}/\mathrm{K} \qquad g \approx 10\mathrm{m/s}^2 \qquad c \approx 10^3\mathrm{J/kg/K} \quad \rightarrow \quad \mathrm{Ge} \approx 10^{-5} H$$

Importance of Φ: non-dimensionalization

- Convection in Earth mantle
- Rotating systems
- Planets with strong gravity or large in size
- Wind farms?
- High-speed (typically: compressible) flows

- “dissipative energy should be included in numerical weather prediction models, particularly in models that resolve mesoscale structures in storms” [1]
- Seems to be included in NCAR model [2]

$$\mathrm{Ge} = \frac{\beta g H}{c}$$



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Effects of viscous dissipation in natural convection

By B. GEBHART

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(Received 8 January 1962)

The effect of viscous dissipation in natural convection is appreciable when the induced kinetic energy becomes appreciable compared to the amount of heat transferred. This occurs when either the equivalent body force is large or when the convection region is extensive. Viscous dissipation is considered here for vertical surfaces subject to both isothermal and uniform-flux surface conditions. A perturbation method is used and the first temperature perturbation function is calculated for Prandtl numbers from 10^{-2} to 10^4. The magnitude of the effect depends upon a dissipation number, which is not expressible in terms of the Grashof or the Prandtl number.

[1] Businger et al., Viscous Dissipation of Turbulence Kinetic Energy in Storms, 2001
[2] Boville et al., Heating and Kinetic Energy Dissipation in the NCAR Community Atmosphere Model, 2003

Non-dimensionalization

- Independent of the choice of u_{ref}, we get

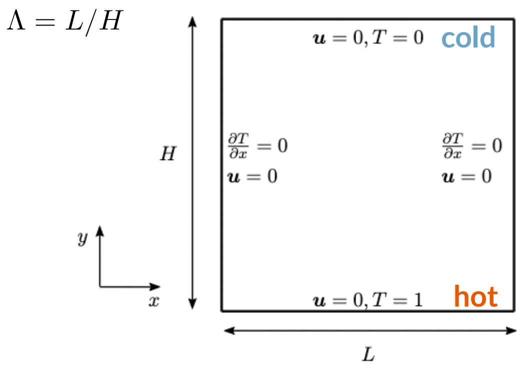
$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\boldsymbol{u}} \otimes \tilde{\boldsymbol{u}}) = -\tilde{\nabla} \tilde{p}' + \alpha_1 \tilde{\nabla}^2 \tilde{\boldsymbol{u}} + \alpha_2 \tilde{T} \boldsymbol{e}_y$$

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\boldsymbol{u}} \tilde{T}) = \alpha_3 \tilde{\Phi} + \alpha_4 \tilde{\nabla}^2 \tilde{T}$$

- Different options for u_{ref} :

	u_{ref}	$\alpha_1 = \frac{\nu}{u_{\mathrm{ref}} H}$	$\alpha_2 = \frac{\beta g \Delta T H}{u_{\mathrm{ref}}^2}$	$\alpha_3 = \frac{\nu u_{\mathrm{ref}}}{c \Delta T H}$	$\alpha_4 = \frac{\kappa}{u_{\mathrm{ref}} H}$	$\gamma = \frac{\alpha_1}{\alpha_3}$
I	$\sqrt{\beta g \Delta T H}$	$\sqrt{\frac{\mathrm{Pr}}{\mathrm{Ra}}}$	1	$\mathrm{Ge}\sqrt{\frac{\mathrm{Pr}}{\mathrm{Ra}}}$	$\frac{1}{\sqrt{\mathrm{PrRa}}}$	$\frac{1}{\mathrm{Ge}}$
II	$\frac{\kappa}{H}$	Pr	PrRa	$\frac{\mathrm{Ge}}{\mathrm{Ra}}$	1	$\frac{\mathrm{PrRa}}{\mathrm{Ge}}$
New? → III	$\sqrt{c \Delta T}$	$\sqrt{\frac{\mathrm{PrGe}}{\mathrm{Ra}}}$	Ge	$\sqrt{\frac{\mathrm{PrGe}}{\mathrm{Ra}}}$	$\sqrt{\frac{\mathrm{Ge}}{\mathrm{PrRa}}}$	1

Non-dimensional total energy



- Choice III gives consistent non-dim. of global internal and kinetic energy

$$\tilde{\nabla} \cdot \tilde{\boldsymbol{u}} = 0$$

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\boldsymbol{u}} \otimes \tilde{\boldsymbol{u}}) = -\tilde{\nabla} \tilde{p}' + \alpha_1 \tilde{\nabla}^2 \tilde{\boldsymbol{u}} + \alpha_2 \tilde{T} \boldsymbol{e}_y$$

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{\nabla} \cdot (\tilde{\boldsymbol{u}} \tilde{T}) = \alpha_3 \tilde{\Phi} + \alpha_4 \tilde{\nabla}^2 \tilde{T}$$

$$\frac{\mathrm{d}\tilde{E}_k}{\mathrm{d}\tilde{t}} = -\frac{\alpha_1}{\Lambda} \int_{\tilde{\Omega}} \tilde{\Phi} \, \mathrm{d}\tilde{\Omega} + \frac{\alpha_2}{\Lambda} \int_{\tilde{\Omega}} \tilde{T} \tilde{v} \, \mathrm{d}\tilde{\Omega}$$

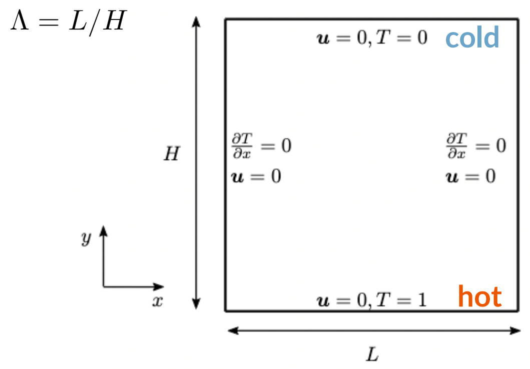
$$\frac{\mathrm{d}\tilde{E}_i}{\mathrm{d}\tilde{t}} = \frac{\alpha_3}{\Lambda} \int_{\tilde{\Omega}} \tilde{\Phi} \, \mathrm{d}\tilde{\Omega} + \frac{\alpha_4}{\Lambda} \int_{\partial\tilde{\Omega}} \tilde{\nabla} \tilde{T} \cdot \boldsymbol{n} \, \mathrm{d}\tilde{S}$$

$$\frac{\mathrm{d}\tilde{E}}{\mathrm{d}\tilde{t}} = \frac{\mathrm{d}\tilde{E}_k}{\mathrm{d}t} + \gamma \frac{\mathrm{d}\tilde{E}_i}{\mathrm{d}t} = \frac{\alpha_2}{\Lambda} \int_{\tilde{\Omega}} \tilde{T} \tilde{v} \, \mathrm{d}\tilde{\Omega} + \frac{\alpha_4}{\Lambda} \int_{\partial\tilde{\Omega}} \tilde{\nabla} \tilde{T} \cdot \boldsymbol{n} \, \mathrm{d}\tilde{S}$$

potential energy flux

heat flux over cold and hot plate

Quantity of interest



- Heat flux on hot and cold plate: **Nusselt number**

$$\mathrm{Nu}(y') := \frac{1}{\Lambda}\int_0^{\Lambda}\left(\frac{1}{\alpha_4}\tilde{T}\tilde{v} - \frac{\partial \tilde{T}}{\partial \tilde{y}}\right)_{(\tilde{x},\tilde{y}')} \mathrm{d}\tilde{x}$$

- From internal energy equation:

$$\alpha_4(\mathrm{Nu}(1) - \mathrm{Nu}(0)) = \alpha_3 \epsilon_U \qquad \epsilon_U := \frac{1}{\Lambda}\int_0^1\int_0^{\Lambda}\tilde{\Phi}\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$$

- **Dissipation causes difference between heat flux on hot and cold plate**

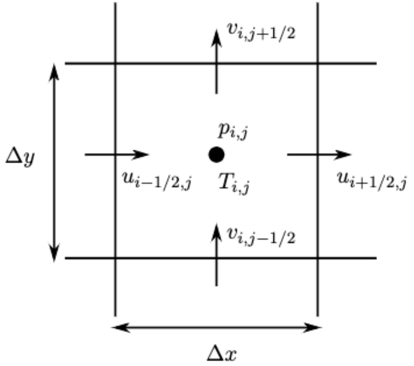
Viscous and thermal dissipation relations

origin	without viscous dissipation	with viscous dissipation
internal	$\mathrm{Nu}(1) = \mathrm{Nu}(0)$	$\alpha_4(\mathrm{Nu}(1) - \mathrm{Nu}(0)) = \alpha_3 \epsilon_U(1)$
kinetic	$\alpha_2\alpha_4(\mathrm{Nu}(0) - 1) = \alpha_1 \epsilon_U(1)$	$\alpha_2\alpha_4(\mathrm{Nu}(0) - 1) = \alpha_1 \epsilon_U(1) - \alpha_2\alpha_3 \int_0^1 \epsilon_U(\tilde{y})\,\mathrm{d}\tilde{y}$
internal energy $\times T$	$\mathrm{Nu}(0) = \epsilon_T$	$\alpha_4 \mathrm{Nu}(0) = \alpha_4 \epsilon_T - \frac{\alpha_3}{\Lambda} \int_{\tilde{\Omega}} \tilde{T}\tilde{\Phi}\,\mathrm{d}\tilde{\Omega}$

Table 2: Steady-state Nusselt number relations, with and without viscous dissipation.

Let’s discretize

Discretization of mass + momentum



- Finite volumes, staggered grid

$$M V_h(t) = 0$$

$$\Omega_V \frac{\mathrm{d}V_h(t)}{\mathrm{d}t} = -C_V(V_h(t)) - G p_h(t) + \alpha_1 D_V V_h(t) + \alpha_2 (A T_h(t) + y_T)$$

- **Implied** kinetic energy equation:

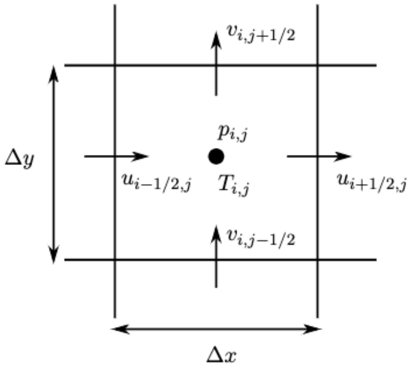
$$\frac{\mathrm{d}E_{k,h}}{\mathrm{d}t} = -\alpha_1 \epsilon_{U,h} + \alpha_2 V_h^T (A T_h + y_T)$$

Global dissipation.
What is the local one?

$$\epsilon_{U,h} = \|Q V_h\|_2^2 > 0$$

(based on skew-symmetry convection; divergence-free velocity; div-grad relation)

Internal energy equation



- Internal energy equation requires approximation to Φ :

$$\Omega_p \frac{\mathrm{d}T_h}{\mathrm{d}t} = -C_T(V_h, T_h) + \alpha_3 \Omega_p \Phi_h(V_h) + \alpha_4 (D_T T_h + \hat{y}_T)$$

- Recall kinetic energy:

$$\frac{\mathrm{d}E_{k,h}}{\mathrm{d}t} = -\alpha_1 \epsilon_{U,h} + \alpha_2 V_h^T (A T_h + y_T)$$

- Φ_h cannot be chosen independently, but is **implied by momentum discretization**. Consistency requirement:

$$\alpha_3 1^T \Omega_p \Phi_h = \alpha_1 \epsilon_{U,h}$$

Energy-consistent discretization of Φ

- How? Choose **a local kinetic energy** (at the pressure point):

$$k_{i,j} := \frac{1}{4}u^2_{i+1/2,j} + \frac{1}{4}u^2_{i-1/2,j} + \frac{1}{4}v^2_{i,j+1/2} + \frac{1}{4}v^2_{i,j-1/2}$$

- Substitute momentum equations, e.g.

$$\frac{\mathrm{d}u_{i+1/2,j}}{\mathrm{d}t} + \mathrm{conv} = -\frac{p_{i+1,j} - p_{i,j}}{\Delta x} + \mu\frac{u_{i+3/2,j} - 2u_{i+1/2,j} + u_{i-1/2,j}}{\Delta x^2} + \mu\frac{u_{i+1/2,j+1} - 2u_{i+1/2,j} + u_{i+1/2,j-1}}{\Delta y^2}$$

- Rewrite viscous terms with discrete version of

$$\int u\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{d}x = \underbrace{\left[u\frac{\mathrm{d}u}{\mathrm{d}x}\right]}_{\text{boundary term}} - \int\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{d}x$$

$$\frac{\mathrm{d}\tilde{E}}{\mathrm{d}\tilde{t}} = \frac{\mathrm{d}\tilde{E}_k}{\mathrm{d}t} + \gamma\frac{\mathrm{d}\tilde{E}_i}{\mathrm{d}t} = \frac{\alpha_2}{\Lambda}\int_{\tilde{\Omega}} \tilde{T}\tilde{v}\,\mathrm{d}\tilde{\Omega} + \frac{\alpha_4}{\Lambda}\int_{\partial\tilde{\Omega}} \tilde{\nabla}\tilde{T}\cdot\boldsymbol{n}\,\mathrm{d}\tilde{S}$$

Energy-consistent discretization of Φ

- **Our new expression** for Φ_h :

$$\boxed{\Phi_{i,j} = \frac{1}{2}\Phi^u_{i+1/2,j} + \frac{1}{2}\Phi^u_{i-1/2,j} + \frac{1}{2}\Phi^v_{i,j+1/2} + \frac{1}{2}\Phi^v_{i,j-1/2}}$$

where

$$\Phi^u_{i+1/2,j} = -\frac{1}{2}\left(\frac{u_{i+3/2,j} - u_{i+1/2,j}}{\Delta x}\right)^2 - \frac{1}{2}\left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x}\right)^2 - \frac{1}{2}\left(\frac{u_{i+1/2,j+1} - u_{i+1/2,j}}{\Delta y}\right)^2 - \frac{1}{2}\left(\frac{u_{i+1/2,j} - u_{i+1/2,j-1}}{\Delta y}\right)^2$$

- **Resulting total energy equation**:

potential energy flux

heat flux over cold and hot plate

$$\frac{\mathrm{d}E_h}{\mathrm{d}t} = \frac{\mathrm{d}E_{k,h}}{\mathrm{d}t} + \gamma\frac{\mathrm{d}E_{i,h}}{\mathrm{d}t} = \alpha_2 V_h^T(AT_h + y_T) + \gamma\alpha_4 1^T(D_T T_h + y_T),$$
$$= \alpha_2 V_h^T(AT_h + y_T) + \gamma\alpha_4(\mathrm{Nu}_H - \mathrm{Nu}_C),$$

Discrete Nusselt number

- Nusselt relations:

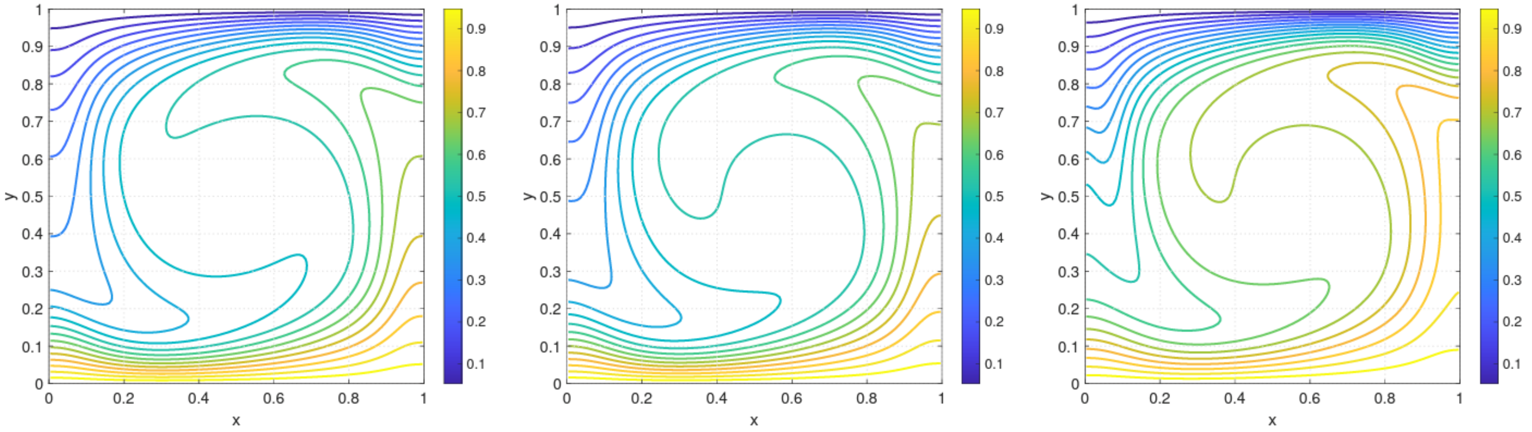
$$\alpha_4(\mathrm{Nu}_C - \mathrm{Nu}_H) = \alpha_3 1^T \Omega_p \Phi_h(V_h)$$

$$\mathrm{Nu}_H := -\sum_{i=1}^{N_x} \frac{T_{i,1} - T_H}{\frac{1}{2}\Delta y} \Delta x \qquad \mathrm{Nu}_C := -\sum_{i=1}^{N_x} \frac{T_C - T_{i,N_y}}{\frac{1}{2}\Delta y} \Delta x$$

- Definition Nusselt number **implied** by discretization diffusion term

Numerical tests

Results – steady state



(a) Ge = 0. (b) Ge = 0.1. (c) Ge = 1.

Results – steady state

- Viscous dissipation does not change the critical Rayleigh number, but leads to a difference between the Nusselt number of the upper and lower plate.

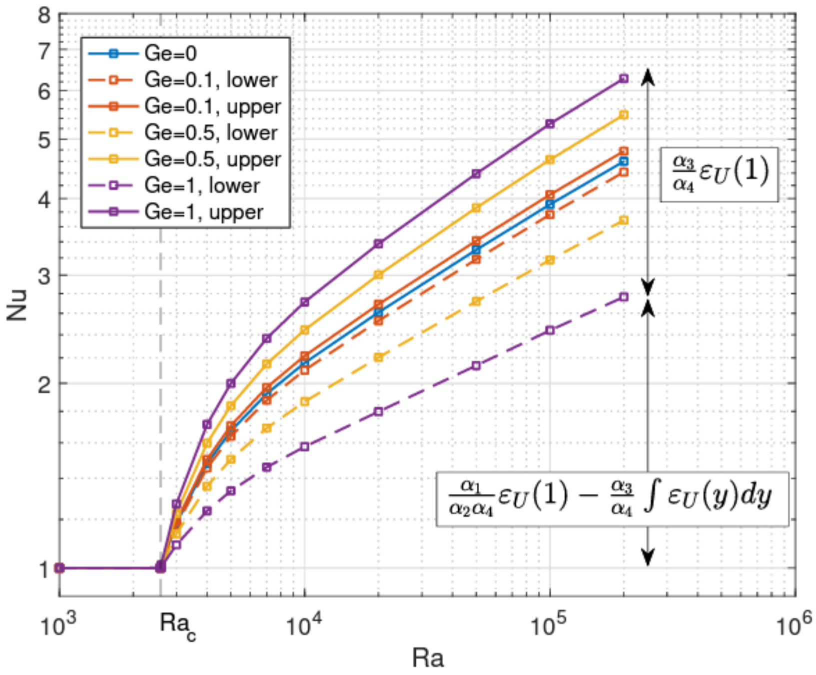


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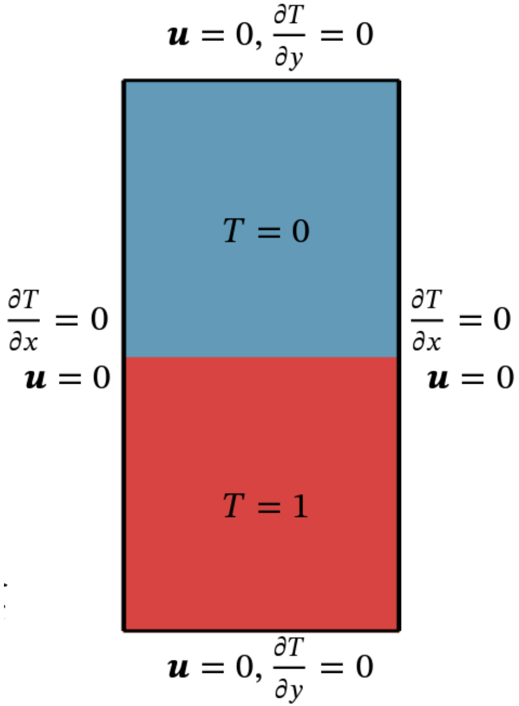
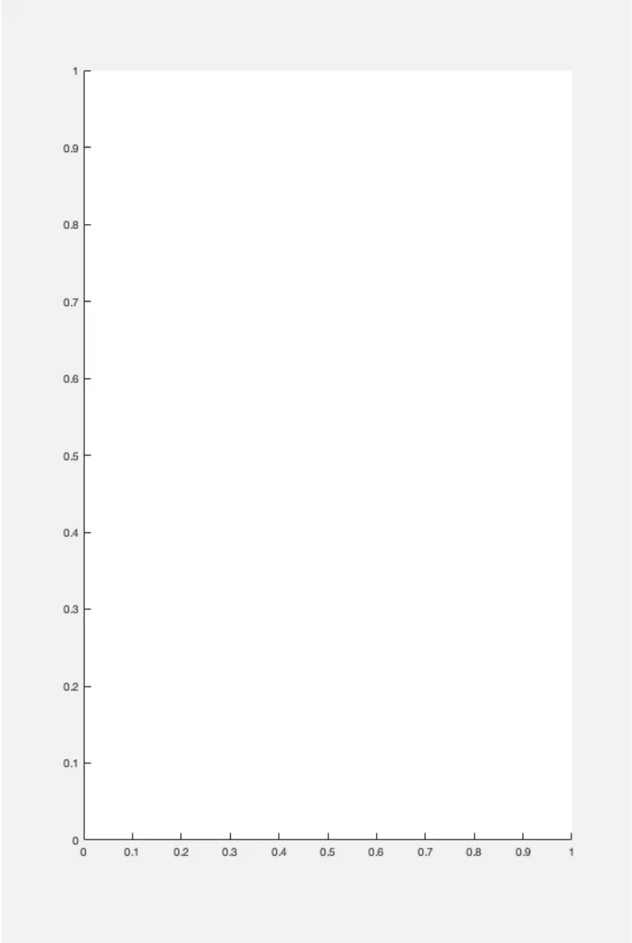
Table 2: Steady-state Nusselt number relations, with and without viscous dissipation.

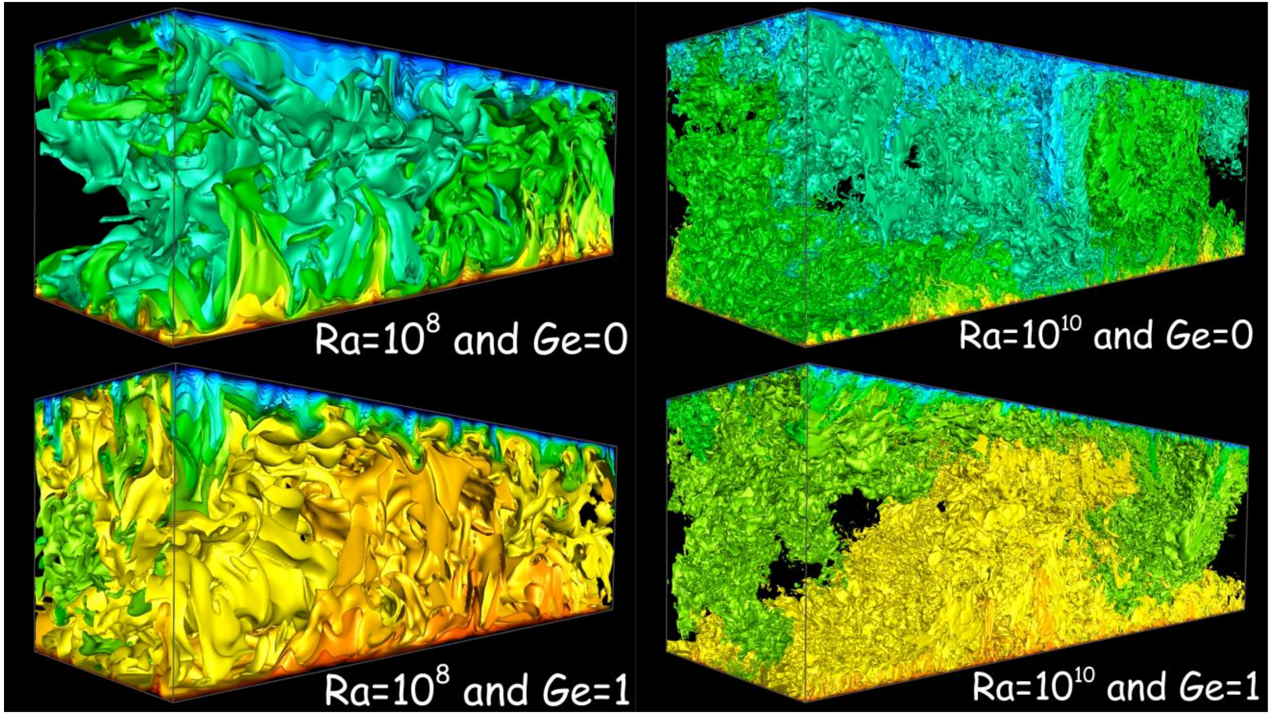
Results – Rayleigh-Taylor

- Exact exchange of kinetic + internal energy vs. potential energy

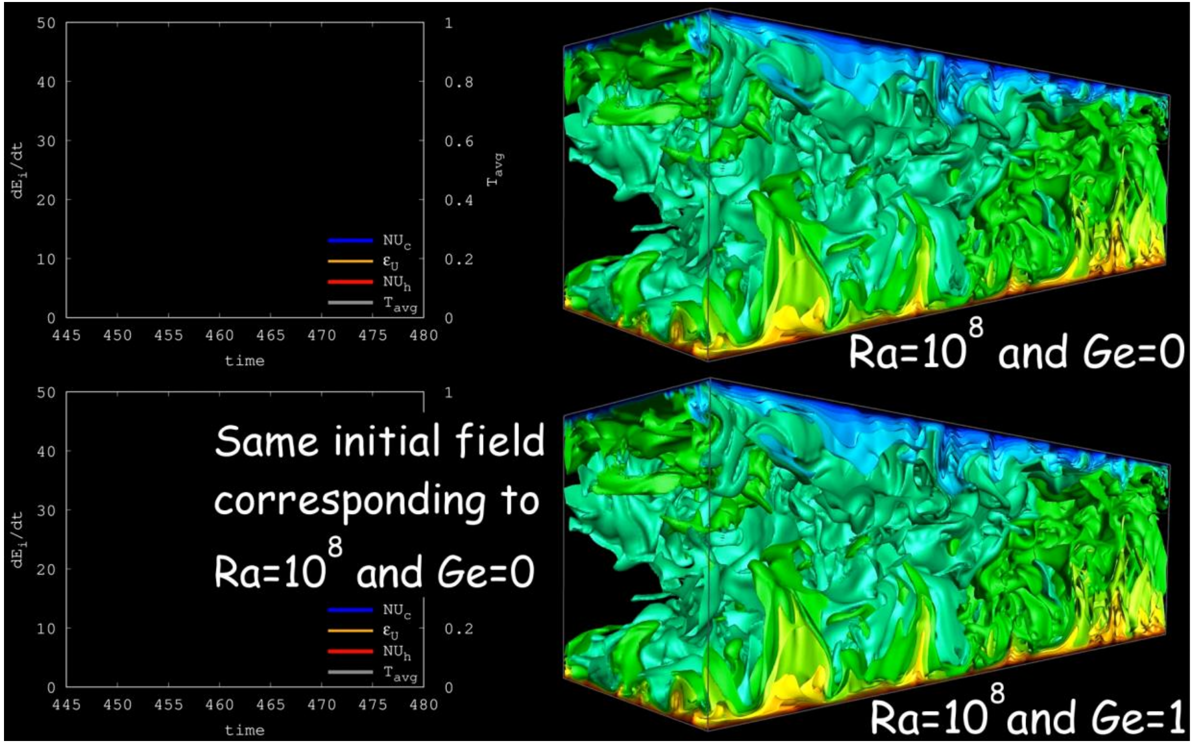




Results – 3D Rayleigh-Bénard

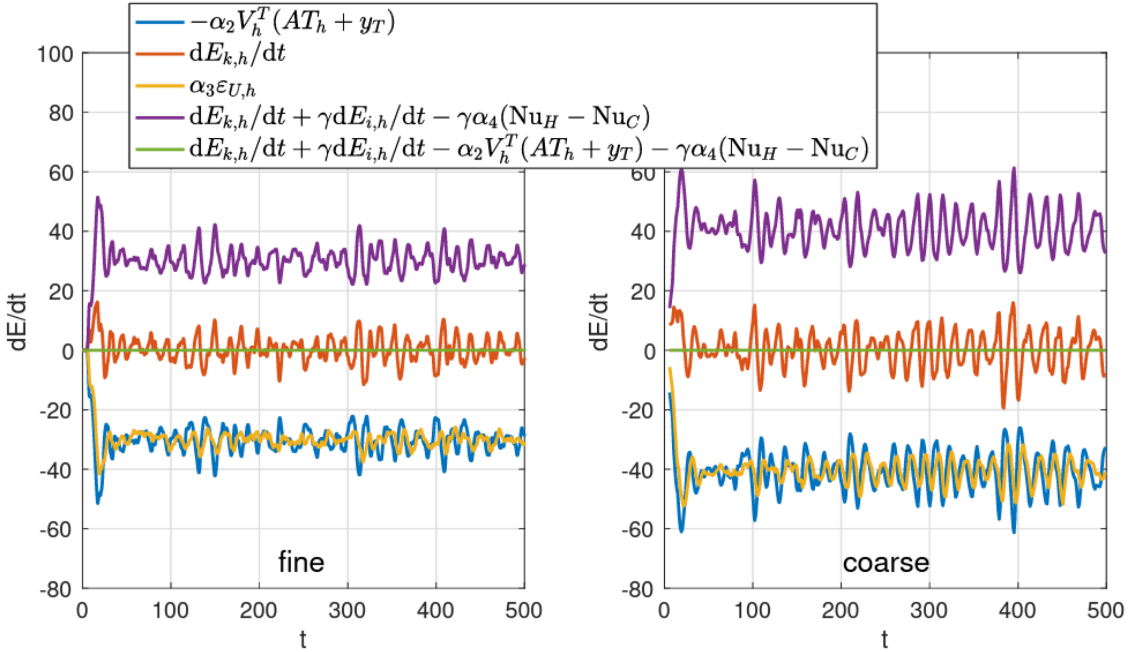


Results – 3D Rayleigh-Bénard



$$\frac{\mathrm{d}E_{k,h}}{\mathrm{d}t} + \gamma \frac{\mathrm{d}E_{i,h}}{\mathrm{d}t} - \alpha_2 V_h^T (AT_h + y_T) - \gamma\alpha_4(\mathrm{Nu}_H - \mathrm{Nu}_C) = 0.$$

Results – 3D Rayleigh-Bénard



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kinetic	$\alpha_2\alpha_4(\mathrm{Nu}(0) - 1) = \alpha_1\epsilon_U(1)$	$\alpha_2\alpha_4(\mathrm{Nu}(0) - 1) = \alpha_1\epsilon_U(1) - \alpha_2\alpha_3 \int_0^1 \epsilon_U(\tilde{y})\,\mathrm{d}\tilde{y}$
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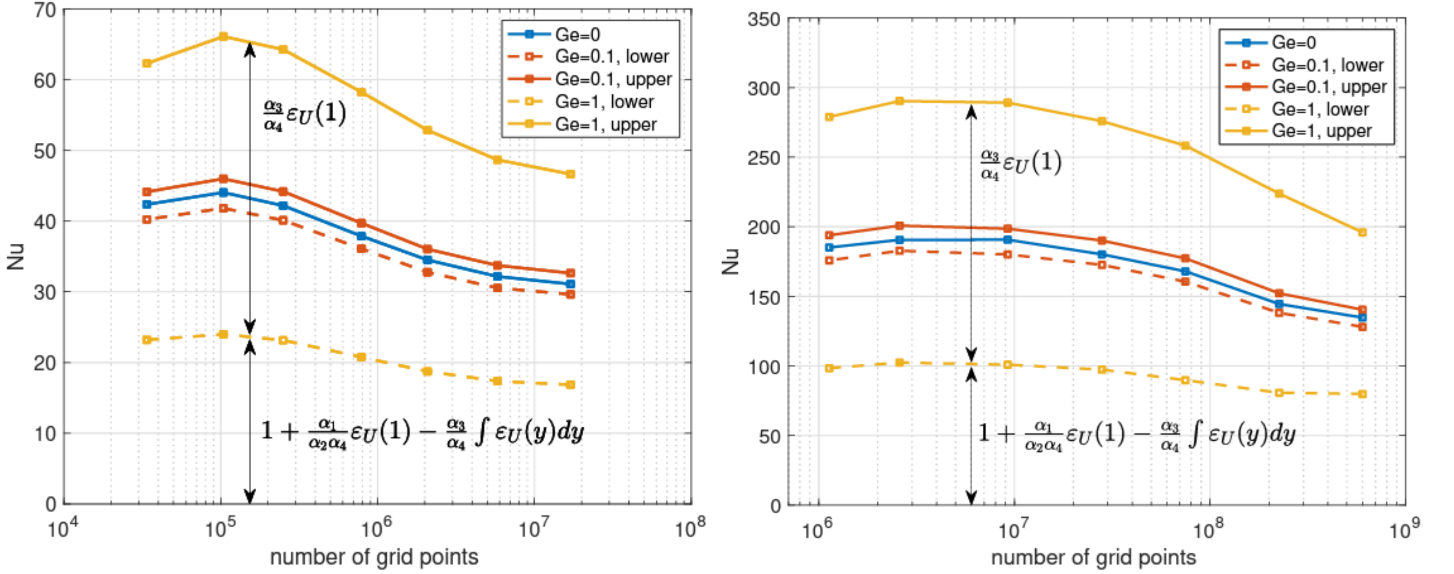
Results – 3D Rayleigh-Bénard



(a) $\mathrm{Ra} = 10^8$.

(b) $\mathrm{Ra} = 10^{10}$.

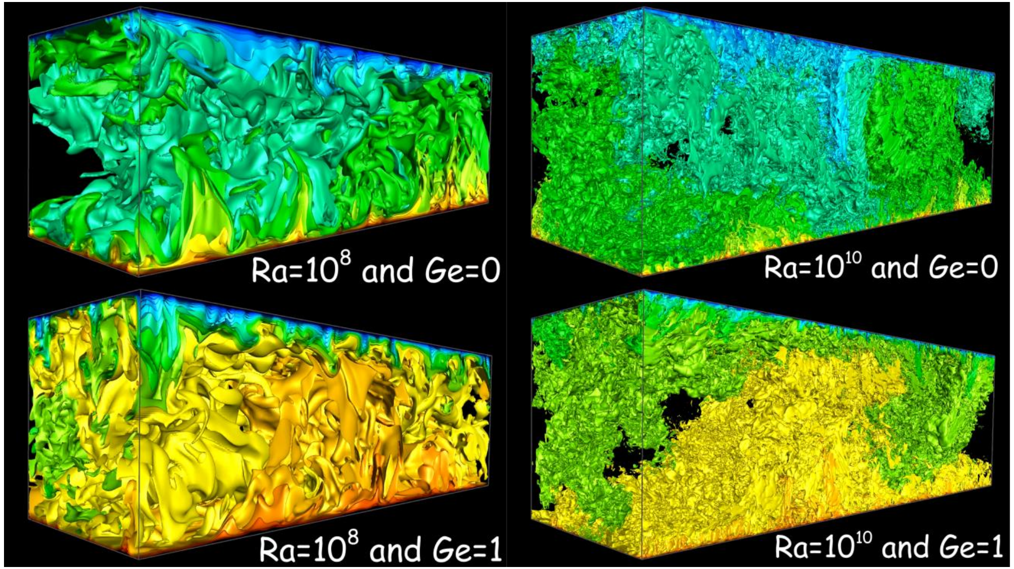
Conclusions

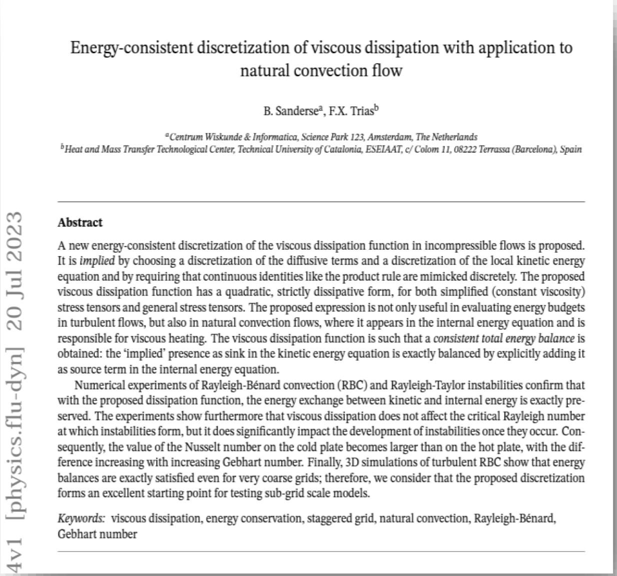
- Energy-consistent discretization of viscous dissipation leads to total energy conservation
- Proposed new non-dimensionalization: consistency momentum & internal energy
- Derived **local discrete kinetic energy** eqn. on staggered grid -> **discrete viscous dissipation term** in internal energy eqn.
- Derived continuous & **discrete Nusselt number** relations that include dissipation
- Excellent starting point for development of **sub-grid scale models**

Paper on ArXiv

- arXiv:2307.10874v1

Thanks for your attention





Energy-consistent discretization of viscous dissipation with application to natural convection flow

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Abstract

A new energy-consistent discretization of the viscous dissipation function in incompressible flows is proposed. It is *implied* by choosing a discretization of the diffusive terms and a discretization of the local kinetic energy equation and by requiring that continuous identities like the product rule are mimicked discretely. The proposed viscous dissipation function has a quadratic, strictly dissipative form, for both simplified (constant viscosity) stress tensors and general stress tensors. The proposed expression is not only useful in evaluating energy budgets in turbulent flows, but also in natural convection flows, where it appears in the internal energy equation and is responsible for viscous heating. The viscous dissipation function is such that a *consistent total energy balance* is obtained: the 'implied' presence as sink in the kinetic energy equation is exactly balanced by explicitly adding it as source term in the internal energy equation.

Numerical experiments of Rayleigh-Bénard convection (RBC) and Rayleigh-Taylor instabilities confirm that with the proposed dissipation function, the energy exchange between kinetic and internal energy is exactly preserved. The experiments show furthermore that viscous dissipation does not affect the critical Rayleigh number at which instabilities form, but it does significantly impact the development of instabilities once they occur. Consequently, the value of the Nusselt number on the cold plate becomes larger than on the hot plate, with the difference increasing with increasing Gebhart number. Finally, 3D simulations of turbulent RBC show that energy balances are exactly satisfied even for very coarse grids; therefore, we consider that the proposed discretization forms an excellent starting point for testing sub-grid scale models.

Keywords: viscous dissipation, energy conservation, staggered grid, natural convection, Rayleigh-Bénard, Gebhart number