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An energy-preserving unconditionally stable fractional step method with high order operators on collocated unstructured grids

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Abstract. In this work, an energy-preserving unconditionally stable fractional step method on collocated unstructured grids is presented. Its formulation is based on preserving the underlying symmetries of the differential operators. This formulation was proven to be unconditionally stable even for highly distorted meshes [1, 2]. However, this formulation was second-order accurate for Cartesian meshes. Within this context, new insights of higher order accurate operators are presented. This high order formulation is based on the linear combination of different control volumes solutions to the Navier-Stokes equations.

1 Introduction

General purpose CFD codes such as OpenFOAM or ANSYS-Fluent rely on a finite-volume (stencil) discretization over unstructured meshes formulation to solve Navier-Stokes equations due to its simplicity. The stencil formulations solve the discretized equations with an algorithm that goes cell by cell, computing the desired quantities. Alternatively, algebraic formulations preserve them in matrix-vector form, and compute the desired quantities by using these matrices and vectors.

A collocated fully-conservative algebraic symmetry-preserving formulation of incompressible Navier-Stokes equations was proposed by Trias et al.[3]. Assuming n control volumes and m faces:

$$\Omega \frac{d\mathbf{u}_c}{dt} + \mathbf{C}(\mathbf{u}_s)\mathbf{u}_c = \mathbf{D}\mathbf{u}_c - \Omega \mathbf{G}_c p_c, \tag{1}$$

$$\mathbf{M}\mathbf{u}_s = \mathbf{0}_c, \tag{2}$$

where $\mathbf{u}_c \in \mathbb{R}^{3n}$ and $\mathbf{p}_c \in \mathbb{R}^n$ are the cell-centered velocity and the cell-centered pressure, respectively. The face-centered quantities, such as $\mathbf{u}_s \in \mathbb{R}^m$ are related to the cell-centered quantities via an interpolation operator $\Gamma_{c\to s} \in \mathbb{R}^{m\times 3n}$:

$$\mathbf{u}_s = \Gamma_{c\to s}\mathbf{u}_c. \tag{3}$$

Finally, $\Omega \in \mathbb{R}^{3n\times 3n}$ is a diagonal matrix containing the cell volumes, $\mathbf{C}(\mathbf{u}_s) \in \mathbb{R}^{3n\times 3n}$ is the discrete

convective operator, $\mathbf{D} \in \mathbb{R}^{3n\times 3n}$ is the discrete diffusive operator, $\mathbf{G}_c \in \mathbb{R}^{3n\times n}$ is the cell-to-cell discrete gradient operator and $\mathbf{M} \in \mathbb{R}^{n\times m}$ is the face-to-cell discrete divergence operator. The velocity correction after applying the Fractional Step Method (FSM) to the Navier-Stokes equations reads:

$$\mathbf{u}_c^{n+1} = \mathbf{u}_c^{p} - \Gamma_{s\to c}\mathbf{G}p_c^{n+1}, \tag{4}$$

where $\Gamma_{s\to c} \in \mathbb{R}^{3n\times m}$ is the face-to-cell interpolator, which is related to the cell-to-face interpolator via the volume matrices $\Gamma_{s\to c} = \Omega^{-1}\Gamma_{c\to s}\Omega_s$, and $\mathbf{G} \in \mathbb{R}^{m\times n}$ is the cell-to-face gradient operator.

Only five discrete operators are needed to formulate these equations: the cell-centered and staggered control volumes (diagonal matrices), Ω_c and Ω_s, the face normal vectors, N_s, the scalar cell-to-face interpolation, $\Pi_{c\to s}$ and the cell-to-face divergence operator, $\mathbf{M}$. For more details of these operators and its construction, the reader is referred to [3]. Due to its simplicity, these operators can be easily builded in existing codes, such as OpenFOAM [4].

2 An energy-preserving unconditionally stable FSM

From our perspective, respecting the symmetries of these differential operators is crutial in order to respect the physical structure of the equations. For example, constructing $\mathbf{G} = -\Omega_s\mathbf{M}^T$ is essential to preserve kinetic energy [3], but it is also mimicking the symmetries of the continuos level operators.

The utility of an algebraic formulation can be found, as an example, in ([1], [2]). In these works, the matrix-vector formulation is used in order to study the stability of the solution in terms of the pressure gradient interpolation in collocated frameworks. To do so, the eigenvalues of $\mathbf{L} - \mathbf{L}_c$ were deeply studied ($\mathbf{L} = \mathbf{MG} \in \mathbb{R}^{n\times n}$ is the compact Laplacian operator whereas $\mathbf{L}_c = \mathbf{M}\Gamma_{c\to s}\Gamma_{s\to c}\mathbf{G} \in \mathbb{R}^{n\times n}$ is the collocated wide-stencil Laplacian operator), and the cell-to-face interpolation that leads to an unconditionally stable FSM turned out to be:

$$\Pi_{c\to s} = \Delta_s^{-1}\Delta_{sc}^T \in \mathbb{R}^{m\times n}, \tag{5}$$

where $\Delta_s \in \mathbb{R}^{m\times m}$ is a diagonal matrix containing the projected distances between two adjacent control volumes, and $\Delta_{sc} \in \mathbb{R}^{n\times m}$ is a matrix containing the projected distance between a cell node and its corresponding face. For details, the reader is referred to [1, 2].

3 Towards a high order formulation

High order formulations are usually preferred over low order ones due to computational power reasons: they have better accuracy for the same mesh resolution and higher arithmetic intensity. A high order (staggered) discretization over structured meshes of Navier-Stokes equations can be found in [5]. This discretization relies on the possibilty of building coarser meshes using bigger control volumes and then eliminate the leading term error by a Richardson extrapolation.

However, this formulation cannot be extended to unstructured meshes due to the impossibility of building these coarser meshes. In this work, preliminary results in constructing high-order operators are presented. This new formulation does not require to build coarser meshes using bigger control volumes, but it is based on a linear combination of the (discrete) solutions of the equations for the different control volumes. Thus, this formulation can also be used in unstructured meshes.

In order to illustrate this idea, the (integrated) divergence of a test function ($u = (Acos(ax+1)sin(by+2), Bsin(ax+3)cos(by+4))$ was computed for three 2D Cartesian and regular control volumes, $i-1 : [x_{i-1}, x_i] \times [y_i, y_{i+1}]$, $i : [x_i, x_{i+1}] \times [y_i, y_{i+1}]$ and $i+1 : [x_{i+1}, x_{i+2}] \times [y_i, y_{i+1}]$, with $x_{j+1} - x_j = y_{j+1} - y_j = h$. Fig. 1 shows the error of one of the control volumes as a function of h. As we can expect, and it is shown in the plot, the order of accuracy of the integrated divergence is 2.

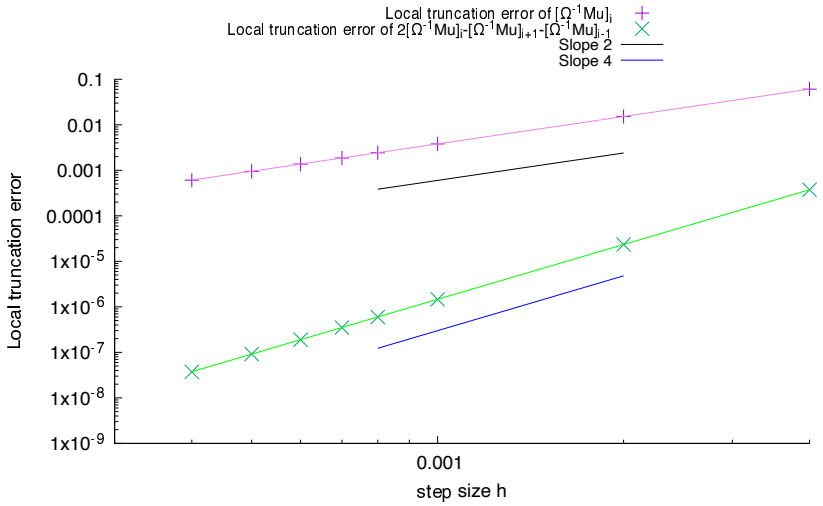

Figure 1: Local truncation error of the divergence operator. Slope for local truncation error of $[\Omega^{-1}Mu]_i$ is 2, while for the linear combination of divergences is 4.

Taking into account that, for the incompressible case, $[\Omega^{-1}\mathbf{Mu}_s]_k = 0$ for every single control volume k, a linear combination of these equations can be used for removing leading error terms, thus the order of accuracy can be reduced to 4. Extension to Navier-Stokes equations is still under studying.

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