On a high-order energy-preserving unconditionally stable discretization on collocated unstructured grids

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2. Towards a high order formulation
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1. Definition of basic collocated operators

Let us suppose we have n control volumes and m faces.

Finite volume discretization of incompressible NS equations on an arbitrary collocated mesh

$$\Omega \frac{d\mathbf{u}_c}{dt} + C(\mathbf{u}_s)\mathbf{u}_c = -D\mathbf{u}_c - \Omega G_c \mathbf{p}_c, \quad (1)$$

$$M\mathbf{u}_s = \mathbf{0}_c. \quad (2)$$

- $\mathbf{p}_c = (p_1, ..., p_n)^T \in \mathbb{R}^n$ is the cell-centered pressure.
- $\mathbf{u}_c = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^T \in \mathbb{R}^{3n}$, where $\mathbf{u}_i = ((u_i)_1, ..., (u_i)_n)^T$ are the vectors containing the velocity components corresponding to the x_i−spatial direction.
- $\mathbf{u}_s = ((u_s)_1, ..., (u_s)_m)^T \in \mathbb{R}^m$ is the staggered velocity.
- The velocities are related via the interpolator from cells to faces $\Gamma_{c\rightarrow s} \in \mathbb{R}^{m\times 3n} \implies \mathbf{u}_s = \Gamma_{c\rightarrow s}\mathbf{u}_c$.

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Definition of basic collocated operators

The (3D) interpolator from cells to faces can be constructed as follows:

$$\Gamma_{c \to s} = N\Pi, \tag{3}$$

where

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- $\Omega_c \in \mathbb{R}^{n \times n}$ is a diagonal matrix with the cell-centered volumes $\implies \Omega = I_3 \otimes \Omega_c$.
- $C_c(\mathbf{u}_s) \in \mathbb{R}^{n \times n}$ is the cell-centered convective operator for a discrete scalar field $\implies C(\mathbf{u}_s) = I_3 \otimes C_c(\mathbf{u}_s)$.
- $D_c \in \mathbb{R}^{n \times n}$ is the cell-centered diffusive operator for a discrete scalar field $\implies D = I_3 \otimes D_c$.

Finally,

- $G_c \in \mathbb{R}^{3n \times n}$ represents the discrete collocated gradient.
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Other useful operators

$$\boxed{G = -\Omega_s^{-1} M^T,}$$
$$L = MG = -M\Omega_s^{-1} M^T,$$
$$L_c = M_c G_c = -M\Gamma_{c\to s}\Omega^{-1}\Gamma_{c\to s}^T M^T,$$
$$\Gamma_{s\to c} = \Omega^{-1}\Gamma_{c\to s}^T \Omega_s. \quad (4)$$

where G is the center-to-face staggered gradient, L is the Laplacian operator, L_c is the collocated-Laplacian operator and $\Gamma_{s\to c}$ is the face-to-cell interpolator.

For more information about Symmetry-Preserving discretization consult: *F.X. Trias, O. Lehmkuhl, A. Oliva, C.D. Perez-Segarra, and R.W.C.P. Verstappen. Symmetry-preserving discretization of Navier-Stokes equations on collocated unstructured meshes. Journal of Computational Physics, 258:246–267, 2014.*

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Conservation of global kinetic energy

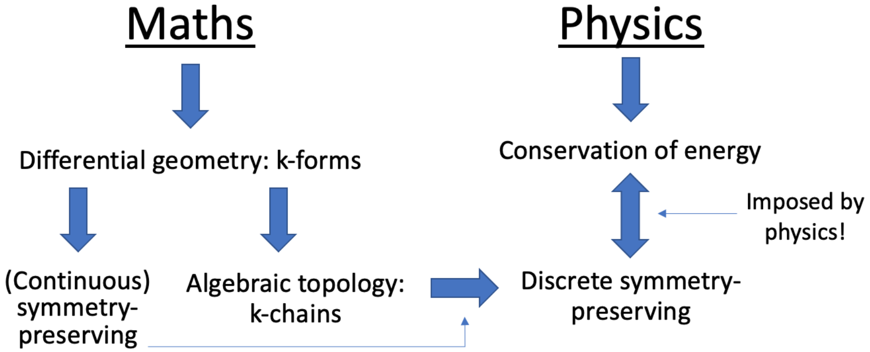
Global kinetic energy equation

$$\frac{d||\mathbf{u}_c||^2}{dt} = -\mathbf{u}_c^T(C(\mathbf{u}_s) + C^T(\mathbf{u}_s))\mathbf{u}_c - \mathbf{u}_c^T(D + D^T)\mathbf{u}_c$$
$$-\mathbf{u}_c^T\Omega G_c\mathbf{p}_c - \mathbf{p}_c^T G_c^T\Omega^T\mathbf{u}_c. \quad (5)$$

In absence of diffusion, that is $D = 0$, the global kinetic energy is conserved if:

- $C(\mathbf{u}_s) = -C^T(\mathbf{u}_s)$, i.e, the convective operator should be skew-symmetric.
- $(-\Omega G_c)^T = M\Gamma_{c\rightarrow s}$, because $M\mathbf{u}_s = \mathbf{0}_c$.

Mimicking continuos properties



A stable pressure gradient interpolation for the velocity correction.

Global kinetic energy equation with skew-symmetric convective operator

$$\frac{d||\mathbf{u}_c||^2}{dt} = -\mathbf{u}_c^T(D + D^T)\mathbf{u}_c - \mathbf{u}_c^T \Omega G_c \mathbf{p}_c - \mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c.$$

In absence of diffusion, that is $D = 0$, the global kinetic energy is conserved if:

- $(-\Omega G_c)^T = M\Gamma_{c \to s}$, because $M\mathbf{u}_s = \mathbf{0}_c$ (But this relation is exact ONLY in staggered configurations!).

In collocated framework, we either solve:

$$Mu_s = 0 \to Lp_c = M\Gamma_{c \to s} u_c^p \to \text{Kinetic Energy Error} \quad (6)$$

$$M_c u_c = 0 \to L_c p_c = M\Gamma_{c \to s} u_c^p \to \text{Checkerboard} \quad (7)$$

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Global kinetic energy equation with skew-symmetric convective operator

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In the first option, the (artificial) kinetic energy added is given by (explicit time integration):

$$-\mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c = \mathbf{p}_c^T (L - L_c)\mathbf{p}_c \Delta t \quad (8)$$

A stable pressure gradient interpolation

- Stable solutions $\rightarrow$ Eigenvalues of $L - L_c$ negative.
- This can be achieved by using the volume weighted scheme:

$$\Pi_{c \rightarrow s} = \Delta_s^{-1} \Delta_{sc}^T, \quad (9)$$

where $\Delta_s \in \mathbb{R}^{m \times m}$ is a diagonal matrix containing the projected distances between two adjacent control volumes, and $\Delta_{sc} \in \mathbb{R}^{m \times n}$ contains the projected distances between an adjacent cell node and its corresponding face.

Volume weighted interpolation: $\phi_f = \frac{V_{s1}}{V_{s1}+V_{s2}} \phi_{c1} + \frac{V_{s2}}{V_{s1}+V_{s2}} \phi_{c2}$.

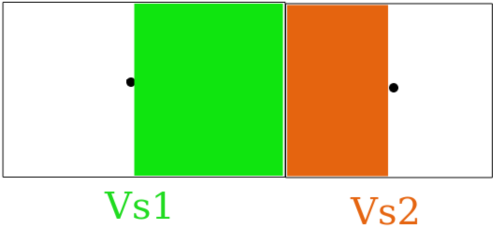

Figure 1: Volume weighted volumes

Summary

We can construct all the previous operators with only five discrete operators:

$$\begin{aligned}\boxed{\Omega_c} &: collocated \;\; volumes\\ \Omega_s &: staggered \;\; volumes\\ N &: face - normal \;\; vectors\\ \Pi_{c\to s} &: Scalar \;\; cell \;\; to \;\; face \;\; interpolation \;\; (''Free'')\\ M &: Divergence \;\; matrix\end{aligned}$$

The divergence operator is usually constructed as follows (discretization of the divergence theorem using the midpoint approximation of the integral):

$$M = T_{sc}S, \tag{10}$$

where T_{sc} is the incidence matrix from faces to cells and S is a diagonal matrix containing face surfaces.

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2. Towards a high order formulation

- Control volume:$[x_i - h/2, x_i + h/2] \times [y_i - h/2, y_i + h/2]$
- Pseudo-control volume:$[x_i - h, x_i + h] \times [y_i - h, y_i + h]$

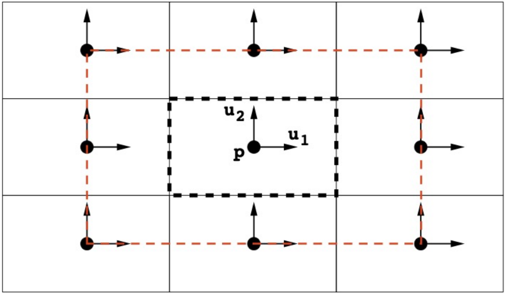


Figure 2: Collocated mesh scheme. Black dashed line shows a typical control volume. Red dashed line shows a pseudo-control volume.

Richardson Extrapolation

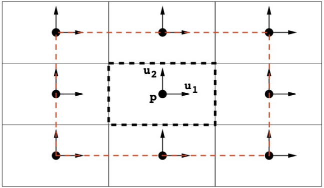
Divergence computed at the control volume and pseudo-control volume respectively:

$$\nabla\cdot = \Omega^{-1}\mathbf{M} + O(h^2), \quad (11)$$

$$\nabla\cdot = \tilde{\Omega}^{-1}\tilde{\mathbf{M}} + 4O(h^2). \quad (12)$$

This notation means $\Omega^{-1}||Mu_s - \int_\Omega \nabla \cdot u||_\infty \sim O(h^2)$ and allows to do linear combinations easily:

$$\nabla\cdot = \frac{4\Omega^{-1}\mathbf{M} - \tilde{\Omega}^{-1}\tilde{\mathbf{M}}}{3} + O(h^4), \quad (13)$$



Richardson Extrapolation: $R(h,t) = \frac{t^n A(h/t) - A(h)}{t^n - 1} = A + O(h^{n+1})$

Richardson Extrapolation

The divergence of a test function ($u = (Acos(ax+1)sin(by+2), Bsin(ax+3)cos(by+4))$ was computed for a Cartesian grid of constant mesh size h:

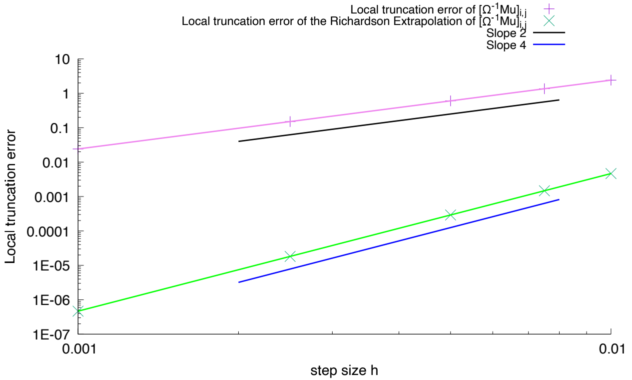


Figure 3: Local truncation error of the discrete divergence **M** operator. Order 2 is found for the discrete divergence and order 4 is found for the Richardson Extrapolation.

Richardson Extrapolation in collocated arrangements

Order 2 is also found for the Richardson Extrapolation. This feature comes from the fact that the interpolation used for computing the velocity at the faces is second order:

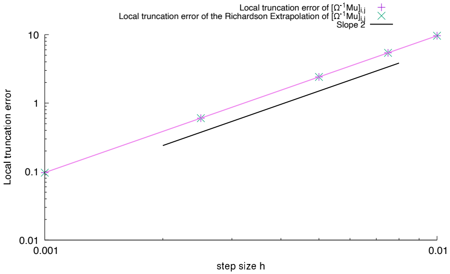


Figure 4: Local truncation error of the discrete divergence **M** operator with collocated variables. Order 2 is found for the discrete divergence and order 2 is found for the Richardson Extrapolation.

Richardson Extrapolation in collocated arrangements

Possible solutions:

- Fourth order interpolation from cells to faces $\rightarrow$ Possible but it can break the symmetries of the operators if it is not done carefully.
- Taking into account that for the first pseudo-control volume, we also know the velocities at the vertex, so we can apply Trapezoidal's rule instead of the mid-point rule $\rightarrow$ it cancels out the divergence operator, so no linear combination is possible.
- Using a second pseudo-control volume of mesh size $4h$ and combining the solutions to eliminate the leading term error.

Double Richardson Extrapolation

Let us assume the divergence of a collocated quantity wants to be calculated at a control volume located at $[x_i - h/2, x_i + h/2] \times [y_i - h/2, y_i + h/2]$. Consider:

- Pseudo-control volume 1:$[x_i - h, x_i + h] \times [y_i - h, y_i + h]$
- Pseudo-control volume 2: $[x_i - 2h, x_i + 2h] \times [y_i - 2h, y_i + 2h]$

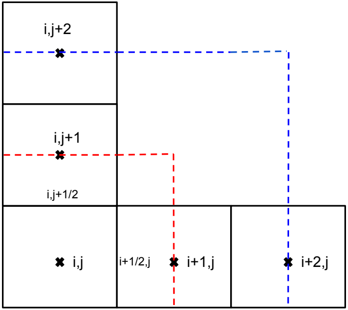


Figure 5: Collocated mesh scheme. Red dashed line shows the first pseudo-control volume and blue dashed line shows the second pseudo-control volume.

Double Richardson Extrapolation

For the same test function as before:

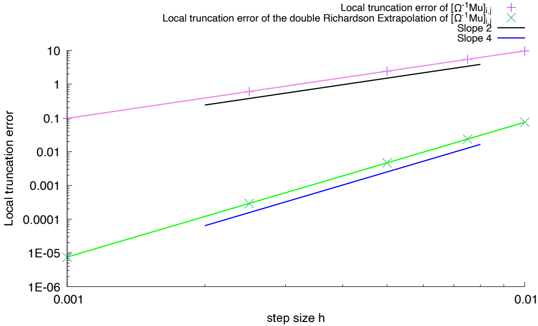


Figure 6: Local truncation error of the discrete divergence **M** operator. Order 2 is found for the discrete divergence and order 4 is found for the double Richardson Extrapolation.

Advantage: Even though same stencil as a high order interpolation is used, no interpolation has to be chosen! $\rightarrow$ The Richardson Extrapolation eliminates directly the low order error.

Algebraic representation of the double Richardson Extrapolation

Generalizing the divergence operator:

$$\mathbf{M} = T_{s\to c}A_f \to \mathbf{M}_{2R} = T_{\bar{c}\to c}A_c. \tag{14}$$

where $T_{s\to c} \in \mathbb{R}^{m\times n}$ is the incidence matrix from faces to cells, $T_{\bar{c}\to c} \in \mathbb{R}^{n\times m\cdot n}$ is an incidence matrix from cells to cells and $A_c \in \mathbb{R}^{m\cdot n\times m\cdot n}$ is a diagonal matrix containing weighted-face areas.

The convective operator can also be rewritten as:

$$\mathbf{C}_c(\mathbf{u}_s)f_c = \mathbf{M}U_f\Pi_{c\to s}f_c \to \tilde{\mathbf{C}}_c(u_s)f_c = \mathbf{M}_{2R}U_f f_f \tag{15}$$

Using the pseudo-control volumes, all the quantities are located at the (pseudo)faces!

Algebraic representation of the double Richardson Extrapolation

For the same test function as before and the new convective operator:

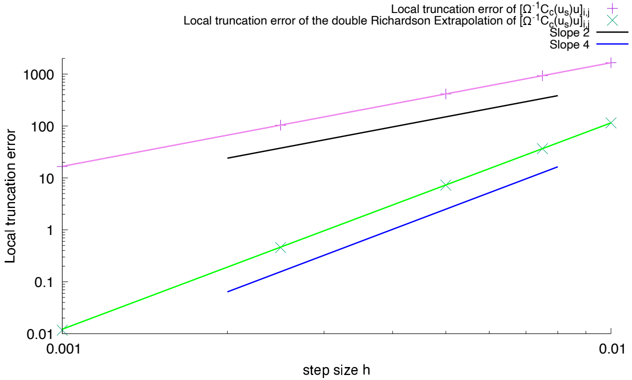


Figure 7: Local truncation error of the discrete collocated convective operator $\mathbf{C}_c(\mathbf{u}_s)$. Order 2 is found for the typical discretization and order 4 is found for the double Richardson Extrapolation.

Possible extension to non-structured meshes

The extension to unstructured meshes can be done by means of the following algorithm:

- Select a control volume i and one of its faces f.
- Select the corresponding neighbour of i regarding f, $neigh(i, f)$ as first pseudo-volume quantity.
- For the second pseudo-volume quantity, compute the maximum of $n_f \cdot n_{neigh(i,f)}$, where n_f is the normal face vector of f and $n_{neigh(i,f)}$. Select the neighbour of this face as the second pseudo-volume quantity.

Doing that, the formulation collapses to the proposed one for the regular Cartesian grid, which is fourth order. No work has been done in this direction, so nothing can be say about the order of convergence for non-regular structured and unstructured grids.

3. Questions and future work

- Is the convective term skew-symmetric? The diagonal term contains the divergence of collocated velocities, but not in the common way.
- If the symmetry-preserving gradient is constructed:

$$\mathbf{G}_{2R} = -\tilde{\Omega}^{-1}\mathbf{M}_{2R}^{T}, \tag{16}$$

  which matrix $\tilde{\Omega}$ of control volumes shall be used?
- How does the coefficients of the Richardson Extrapolation change for non-uniform Cartesian meshes?
- May this collocated high order extension be seen as the linear combination of two (staggered) Navier-Stokes discretizations at the pseudo-control volumes? This will allow to reinterpret the operators easily.

General conclusions

- An energy-preserving unconditionally stable fractional step method on collocated grids has been presented.
- A double Richardson Extrapolation allows to eliminate the leading term error for the divergence and the convective operators in collocated arrangements.
- The symmetry-preserving gradient construction is not straightforward, however, reinterpreting the double Richardson Extrapolation as the solution of two NS equation systems at the pseudo-control volumes may be useful.