A MIMETIC FINITE VOLUME METHOD ON COLLOCATED GRIDS FOR INCOMPRESSIBLE FLOWS

D. Santos, J.A. Hopman, J. Plana, F.X. Trias

Heat and Mass Transfer Technological Center - Technical University of Catalonia

daniel.santos.serrano@upc.edu

1 Introduction

Many of the most used codes, such as OpenFOAM or Fluent, adopt a collocated Finite Volume Method (see Trias et al (2014) and Fig. 1) arrangement due to its simplicity, flexibility, and ease of handling complex geometries. The downsides of this choice over staggered methods (see Verstappen and Veldman (2002)) are well-known, namely, checkerboarding problems (see Hopman et al (2025)) and/or stability issues (see Santos et al (2025)). A Compact Laplacian is usually used to overcome checkerboarding problems. However, it introduces some (artificial) kinetic energy errors that can lead to instabilities. For staggered configurations, respecting the underlying symmetries of the differential operators is sufficient to conserve kinetic energy (see Verstappen and Veldman (2025)). However, in collocated symmetry-preserving methods, there is still an (artificial) contribution to the kinetic energy given by the pressure term if a Compact Poisson is solved for the velocity.

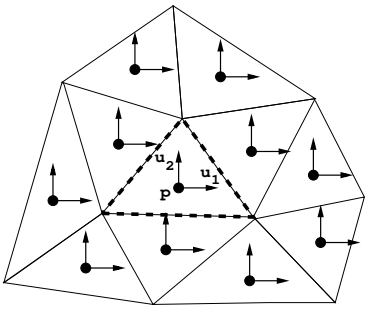

Figure 1: General collocated arrangement.

When overcoming stability problems, many people try to change the interpolations or refine/improve the mesh until the problem is solved. However, this approach may lead to a loss of resources and time. This work intends to study the foundations of the FVM from a general approach, giving the direct link with the 'classical' vector calculus approach, identifying the source of the (artificial) kinetic energy error, and proposing a solution.

Many numerical methods, in particular the FVM, rely on integrating the continuous equations (obtaining integral quantities) and then approximating these integrals in order to obtain a discrete system of equations that can be solved. Finally, once the solution is obtained for a finite set of points, it is extended to the whole domain by some assumption. This process can be studied using the Reduction $\mathcal{R}$ and the Reconstruction $\mathcal{I}$ maps. The former provides the integral quantities, and the latter extends the solution to the whole domain (see Fig. 2).

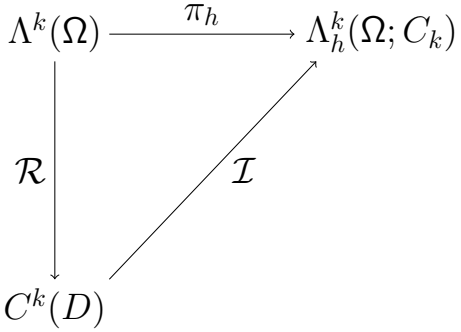

Figure 2: Diagram illustrating the Reduction and Reconstruction maps.

The $\Lambda^k(\Omega)$ space is the space of continuous variables, $C^k(D)$ is the space of integrated quantities, and $\Lambda_h^k(\Omega; C_k)$ is the space of reconstructed variables. The superindex k indicates if the quantity is related to points ($k = 0$), to lines ($k = 1$), to surfaces ($k = 2$) or to volumes ($k = 3$).

2 Reduction and Reconstruction maps

The Reduction and Reconstruction maps need to satisfy the following properties:

1. $\mathcal{R}\mathcal{I} = id$ (consistency property).
2. $\mathcal{I}\mathcal{R} = id + O(h^s)$ (approximation property).

It can happen that $\Lambda_h^k(\Omega; C_k) \notin \Lambda^k(\Omega)$. This is the case for non-continuous Reconstructions, such as FVM or Discontinuous Galerkin methods.

Supposing we have a quantity associated to volumes, $\Phi^{(3)}$, it can be integrated in a control volume, defining the Reduction map, as follows.

$$\int_{\Omega_i} \Phi^{(3)} dV \approx \Phi_i V_i \implies \mathcal{R}\Phi^{(3)} = \Phi_i V_i, \qquad (1)$$

where the approximation of the integral has been done by the Mid Point integral rule (such as typically done

in FVM), Φ_i is the value of Φ at the cell center and V_i is the volume of the control volume i.

Similary, if we have a quantity associated to faces, for example a flux $u^{(2)}$, the Reduction map for 2-quantities can be defined as:

$$\int_f u^{(2)} \mathbf{dS} \approx \mathbf{u}_f \cdot \mathbf{n}_f S_f \implies \mathcal{R}u^{(2)} = \mathbf{u}_f \cdot \mathbf{n}_f S_f, \tag{2}$$

where again the approximation of the integral has been done by means of the Mid Point integral rule, $\mathbf{u}_f$ is the vector velocity at the face center, $\mathbf{n}_f$ is the normal vector to the face and S_f is the surface of the face f.

Finally, we need to define the Reconstruction $\mathcal{I}$, which needs to satisfy the consistency and the approximation property. We can define, for FVM, the Reconstruction $\mathcal{I}$ as:

$$\mathcal{I}(\phi_i V_i) = \phi_i \mathbb{1}_{\Omega_i}(\mathbf{x}), \tag{3}$$

where $\mathbb{1}_{\Omega_i}$ is 1 if $x \in \Omega_i$ and 0 otherwise. The extension to the whole domain is simply:

$$\mathcal{I}\mathcal{R}\Phi^{(3)} = \sum_i \phi_i \mathbb{1}_{\Omega_i}(\mathbf{x}). \tag{4}$$

This is a piecewise constant Reconstruction stating that the value is constant inside each control volume, with discontinuities at the faces of the cells. To extend the values to the faces, FVM uses interpolations.

3 Discretizing operators

Note that the Reduction map is able to provide discretized versions of the continuous operators, for example for the divergence:

$$\mathcal{R}(\nabla \cdot u^{(2)}) = \int_{V_i} \nabla \cdot u^{(2)} dV = \int_{\partial V_i} u^{(2)} \mathbf{dS} \implies \sum_{f \in \partial V_i} \int_f u^{(2)} \mathbf{dS} = \sum_{f \in \partial V_i} \mathcal{R}_f u^{(2)}, \tag{5}$$

where $\mathcal{R}_f u^{(2)}$ is the Reduction of $u^{(2)}$ on face f. Note that this form of discretizing the divergence automatically separates the geometrical operation, which is the one that couples the variable to the geometry (the Reduction map), from the topological part of the operator, which is the summation. Note that this summation is simply the incidence matrix from faces to cells. If we define the incidence matrix from faces to cells as δ, then the discretized divergence becomes

$$\mathcal{R}(\nabla \cdot u^{(2)}) = \delta \mathcal{R} u^{(2)}. \tag{6}$$

By defining the L^2-inner product as

$$(\Phi^{(k)}, \Psi^{(k)})_\Omega := \int_\Omega \langle \Phi^{(k)}, \Psi^{(k)} \rangle dV, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ is the point-wise product, we can compute the metric of the collocated space:

$$\int_\Omega \langle \Phi^{(k)}, \Psi^{(k)} \rangle dV = \sum_i \Phi_i V_i \Psi_i \implies (\Phi^{(k)}, \Psi^{(k)})_\Omega = \Phi_c^T \Omega_c \Psi_c, \tag{8}$$

where Φ_c and Ψ_c are vectors containing the collocated components, and Ω_c is a diagonal matrix containing the collocated volumes in the diagonal. If the vectors implied in the L^2-inner product contain x, y and z components, then

$$(\Phi^{(k)}, \Psi^{(k)})_\Omega = \Phi_c^T \Omega \Psi_c, \tag{9}$$

where $\Omega = \mathbb{I}_3 \otimes \Omega_c$.

The Hilbert adjoint operator of the divergence respecting the L^2-inner product is the gradient, so for the Reconstructed versions:

$$(\mathcal{I}(\delta \mathcal{R} u^{(2)}), \Phi^{(3)})_\Omega = (u^{(2)}, \mathsf{G}\Phi^{(3)})_{\Omega_s} \implies \mathsf{G} = -\Omega_s^{-1} \mathsf{M}^T, \tag{10}$$

where $\mathsf{M} = \delta \mathcal{R}$ is the reduced divergence. Finally, a (Compact) Laplacian can be computed as

$$\mathsf{L} = \mathsf{M}\mathsf{G}. \tag{11}$$

4 Incompressible Navier-Stokes discretization in Collocated grids

Assuming there are n control volumes and m faces, the reduced incompressible Navier-Stokes equations reads:

$$\Omega \frac{d\mathbf{u}_c}{dt} + \mathsf{C}(\mathbf{u}_s)\mathbf{u}_c + \mathsf{D}\mathbf{u}_c + \Omega \mathsf{G}_c \mathbf{p}_c = \mathbf{0}, \tag{12a}$$

$$\mathsf{M}\mathbf{u}_s = \mathbf{0}, \tag{12b}$$

where $\mathbf{p}_c = (p_1, p_2, \ldots, p_n)^T \in \mathbb{R}^n$ and $\mathbf{u}_c \in \mathbb{R}^{3n}$ are the cell-centered pressure and collocated velocity fields, respectively. The subindices c and s indicate if the variables are cell-centered or staggered at the faces. To verify mass conservation within each control volume, a velocity field is defined at the faces $\mathbf{u}_s = \Big((u_s)_1, (u_s)_2, (u_s)_3, \ldots, (u_s)_m\Big)^T \in \mathbb{R}^m$. Quantities defined at cells and at faces are related using the $3-$dimensional interpolator from cells to faces $\Gamma_{c \to s}$, constructed as follows:

$$\Gamma_{c \to s} = N(\mathsf{I}_3 \otimes \Pi_{c \to s}), \tag{13}$$

where $\Pi_{c \to s} \in \mathbb{R}^{m \times n}$ is the scalar cell-to-face interpolator, and $N = (N_{s,x} N_{s,y} N_{s,z}) \in \mathbb{R}^{3m \times m}$, where $N_{s,i} \in \mathbb{R}^{m \times m}$ is a diagonal matrix containing the x_i spatial components of the face normal vectors.

The matrices $\Omega \in \mathbb{R}^{3n \times 3n}$, $\mathsf{C}(\mathbf{u}_s) \in \mathbb{R}^{3n \times 3n}$ and $\mathsf{D} \in \mathbb{R}^{3n \times 3n}$ are block diagonal matrices given by

$$\Omega = \mathsf{I}_3 \otimes \Omega_c, \qquad \mathsf{C}(\mathbf{u}_s) = \mathsf{I}_3 \otimes \mathsf{C}_c(\mathbf{u}_s), \qquad \mathsf{D} = \mathsf{I}_3 \otimes \mathsf{D}_c,$$

where $\mathsf{I}_3 \in \mathbb{R}^{3\times3}$ is the identity matrix and $\Omega_c \in \mathbb{R}^{n\times n}$ is a diagonal matrix containing the cell-centered control volumes. $\mathsf{C}_c(\mathbf{u}_s) \in \mathbb{R}^{n\times n}$ and $\mathsf{D}_c \in \mathbb{R}^{n\times n}$ are the cell-centered convective and diffusive operators for a discrete scalar field, respectively. Finally, $\mathsf{G}_c \in \mathbb{R}^{3n\times n}$ is the discrete gradient operator, and the matrix $\mathsf{M} \in \mathbb{R}^{n\times m}$ is the face-to-center discrete divergence operator.

These discretized equations, when using a projection method such as PISO or the Fractional Step Method (FSM) to decouple the pressure from the velocity, along with a Compact Poisson, are not always stable. The (artificial) kinetic energy term added by the pressure gradient in explicit time integration schemes will be proved to be:

$$\mathbf{p}_c^T(\mathsf{L}-\mathsf{L}_c)\mathbf{p}_c \tag{14}$$

The requirements for the discretization in order to produce stable simulations were identified in the next theorem:

Theorem Assumptions:

- Our projection method adds a kinetic energy error of the form $\mathbf{p}_c^T(\mathsf{L}-\mathsf{L}_c)\mathbf{p}_c^T$ (such as the FSM or PISO).
- The method preserves the symmetries of the differential operators.

Then, $\mathbf{p}_c^T(\mathsf{L}-\mathsf{L}_c)\mathbf{p}_c^T \leq 0$ at each time step $\iff$

1. The volume-weighted interpolator is used for the pressure gradient.
2. $$V_k = \sum_{f\in F(k)} \tilde{V}_{k,f} n_{i,f}^2, \quad \forall k \in \{1,...,n\},$$
$$\sum_{f\in F(k)} \tilde{V}_{k,f} n_{i,f} n_{j,f} = 0, \quad \forall k \in \{1,...,n\},$$

where $\tilde{V}_{k,f} = \delta_{k,f} S_f$, where $\delta_{k,f}$ is the projected distance between the cell center and the face. The volume-weighted interpolator is an interpolation that conserves integrated quantities when interpolating from cells to faces and can be constructed in any mesh as follows:

$$\Pi_{c\to s} = \Delta_s^{-1}\Delta_{sc}^T, \tag{15}$$

where $\Delta_s \in \mathbb{R}^{m\times m}$ is a diagonal matrix containing the projected distances between two adjacent control volumes, and $\Delta_{sc} \in \mathbb{R}^{m\times n}$ is a matrix containing the projected distances between an adjacent cell node and its corresponding face. Fig.3 (right) shows a representation of these distances.

A numerical test in a Turbulent Channel Flow at $Re_\tau = 395$ will be provided to show the stability of the method. To do so, a high distorted mesh will be used (see Fig. 4)

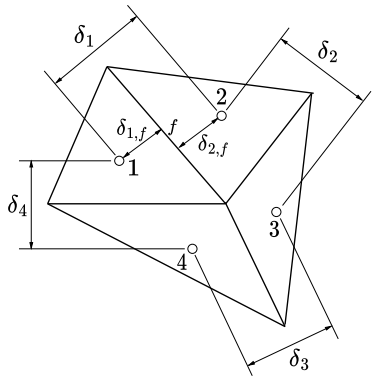

Figure 3: δ_i are the components of Δ_s, while the components of Δ_{sc} would be calculated in the same way but taking the distance between a control volume and their corresponding face centers.

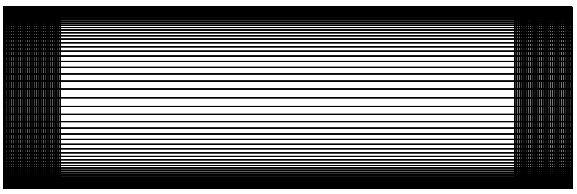

Figure 4: Mesh with a maximum volume ratio between adjacent cells of 250.

Acknowledgments

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