A RATIONAL LENGTH SCALE FOR LARGE-EDDY SIMULATIONS ON ANISOTROPIC GRIDS

F.X.Trias[1], J.Ruano[1], E.Di Lavore[1], A.Duben[2], A.Gorobets[2]

[1] **_Heat and Mass Transfer Technological Center, Technical University of Catalonia, c/Colom 11, 08222 Terrassa, Spain_**
[2] **_Keldysh Institute of Applied Mathematics, 4A, Miusskaya Sq., Moscow 125047, Russia_**
francesc.xavier.trias@upc.edu

1 Introduction

Large-eddy simulation (LES) equations result from applying a spatial commutative filter, with filter length δ, to the Navier–Stokes equations

$$\partial_t \overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} \cdot \nabla)\, \overline{\boldsymbol{u}} = \nu \nabla^2 \overline{\boldsymbol{u}} - \nabla \overline{p} - \nabla \cdot \tau(\overline{\boldsymbol{u}}), \quad \nabla \cdot \overline{\boldsymbol{u}} = 0, \tag{1}$$

where $\overline{\boldsymbol{u}}$ is the filtered velocity and $\tau(\overline{\boldsymbol{u}})$ is the subgrid stress (SGS) tensor and aims to approximate the effect of the under-resolved scales, $\tau(\overline{\boldsymbol{u}}) \approx \overline{\boldsymbol{u} \otimes \boldsymbol{u}} - \overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}$. Because of its inherent simplicity and robustness, the eddy-viscosity assumption is by far the most used closure model, *i.e.* $\tau(\overline{\boldsymbol{u}}) \approx -2\nu_t S(\overline{\boldsymbol{u}})$. Then, the eddy-viscosity, ν_t, is usually modeled as follows

$$\nu_t = (C_m \delta)^2 D_m(\overline{\boldsymbol{u}}). \tag{2}$$

In the last decades, most of the research has focused on either the calculation of the model constant, C_m (*e.g.* the dynamic modeling approach and its variants), or the development of more appropriate model operators $D_m(\overline{\boldsymbol{u}})$ (*e.g.* WALE, Vreman's, Verstappen's, σ-model, S3PQR [1],...). Surprisingly, little attention has been paid on the computation of the subgrid characteristic length, δ. Due to its simplicity and applicability to unstructured meshes, the approach proposed by Deardorff [2], *i.e.* the cube root of the cell volume (see Eq. 4), is by far the most widely used to computed the δ, despite in some situations it may provide very inaccurate results.

Alternative methods to compute δ are summarized and classified in Table 1 according to a list of desirable properties for a (proper) definition of δ. According to the property **P2**, they can be classified into two main families; namely, (i) definitions of δ that solely depend on geometrical properties of the mesh, and (ii) definitions of δ that are also dependent on the local flow topology. The latter is characterized by the gradient of the resolved velocity field, $\mathsf{G} \equiv \nabla \overline{\boldsymbol{u}}$, whereas the local mesh geometry for a Cartesian grid is contained in the following second-order diagonal tensor,

$$\Delta \equiv \mathrm{diag}(\Delta x, \Delta y, \Delta z). \tag{3}$$

Hereafter, we take $\Delta x \le \Delta y \le \Delta z$ without loss of generality. The first seven definition listed in Table 1

	δ_{vol}	δ_{Sco}	$\delta_{\max}$	$\delta_{\boldsymbol{\omega}}$	$\tilde{\delta}_{\boldsymbol{\omega}}$	δ_{SLA}	δ_{lsq}	δ_{rls}	$\tilde{\delta}_{\mathrm{rls}}$
Ref	[2]	[3]	[6]	[7]	[5]	[4]	[8]		
Eq.	(4)	(4)	(5)	(6)	(7)	(7)	(8)	(15)	(18)
P0	Y	Y	Y	Y	Y	Y	Y	Y	Y
P1	Y	Y	Y	Y	Y	Y	Y	Y	Y
P2	N	N	N	Y	Y	Y	Y	Y	Y
P3	Y	N	N	N*	Y	Y	Y	Y	Y
P4	+++	++	++++	++	+	+	+++	++++	+++

Table 1: Properties of different definition of the subgrid characteristic length, δ. Namely, **P0**: $\delta \ge 0$, locality and frame invariant; **P1**: boundedness, *i.e.*, given a structured Cartesian mesh where $\Delta x \le \Delta y \le \Delta z$, $\Delta x \le \delta \le \Delta z$; **P2**: sensitive to flow orientation; **P3**: applicable to unstructured meshes; **P4**: well-conditioned and low computational cost.

are given by

$$\delta_{\mathrm{vol}} = (\Delta x \Delta y \Delta z)^{1/3}, \quad \delta_{\mathrm{Sco}} = f(a_1, a_2)\delta_{\mathrm{vol}}, \tag{4}$$

$$\delta_{\max} = \max(\Delta x, \Delta y, \Delta z), \tag{5}$$

$$\delta_{\boldsymbol{\omega}} = \sqrt{\frac{\omega_x^2 \Delta y \Delta z + \omega_y^2 \Delta x \Delta z + \omega_z^2 \Delta x \Delta y}{|\boldsymbol{\omega}|^2}}, \tag{6}$$

$$\tilde{\delta}_{\boldsymbol{\omega}} = \max_{n,m} \frac{|\boldsymbol{l}_n - \boldsymbol{l}_m|}{\sqrt{3}}, \; \delta_{\mathrm{SLA}} = \tilde{\delta}_{\boldsymbol{\omega}} F_{\mathrm{KH}}(VTM) \tag{7}$$

$$\delta_{\mathrm{lsq}} = \sqrt{(\mathsf{G}_\delta \mathsf{G}_\delta^T : \mathsf{G}\mathsf{G}^T)/(\mathsf{G}\mathsf{G}^T : \mathsf{G}\mathsf{G}^T)}, \tag{8}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the vorticity and $f(a_1, a_2) = \cosh \sqrt{4/27[(\ln a_1)^2 - \ln a_1 \ln a_2 + (\ln a_2)^2]}$ is the correcting function proposed by Scotti [3]. The function $0 \le F_{\mathrm{KH}}(VTM) \le 1$ was proposed by Shur *et al.* [4] to correct the $\tilde{\delta}_{\boldsymbol{\omega}}$ definition proposed by Mockett *et al.* [5], both in the context of Detached Eddy Simulation. Finally, $\mathsf{G}_\delta \equiv \mathsf{G}\Delta$ is the gradient in the so-called computational space. These properties are based on physical, numerical, and/or practical arguments. The list is completed with the novel definitions δ_{rls} and $\tilde{\delta}_{\mathrm{rls}}$ introduced in the next sections.

2 Finite-volume filtering

Let us consider a generic 1D convection-diffusion equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial (u\phi)}{\partial x} = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right), \tag{9}$$

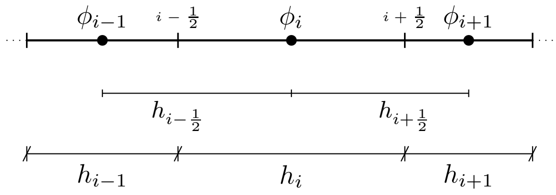

Figure 1: One-dimensional mesh

$$
\begin{tikzcd}
\boldsymbol{\nu}_{t,c} \arrow[r, dashed, "\Omega_c^{-2/3}"] \arrow[d, leftrightarrow, "\mathrm{diag}"'] & \hat{\boldsymbol{\nu}}_{t,c} \arrow[r, "\Pi"] \arrow[d, leftrightarrow, "\mathrm{diag}"'] & \hat{\boldsymbol{\nu}}_{t,s} \arrow[r, dashed, "\Delta_s^2"] \arrow[d, leftrightarrow, "\mathrm{diag}"'] & \boldsymbol{\nu}_{t,s} \arrow[d, leftrightarrow, "\mathrm{diag}"'] \\
\Lambda_c \arrow[r, dashed] & \tilde{\Lambda}_c \arrow[r] & \tilde{\Lambda}_s \arrow[r, dashed] & \Lambda_s
\end{tikzcd}
$$

Figure 2: Flowchart for the implementation of the δ_{rls}. Dashed lines represent the modifications required respect to the usual flowchart.

where $u(x,t)$ denotes the advective velocity and $\phi(x,t)$ represents a generic (transported) scalar field. In a finite-volume method (FVM), this equation is integrated over a set of non-overlapping volumes. Hence, FVM variables result by applying a box filter with filter width equal to the local grid size, h,

$$\overline{\phi}(x) = \frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} \phi dx. \tag{10}$$

Notice that this filter commutes with differentiation

$$\frac{\partial \overline{\phi}}{\partial x} = \frac{1}{h}\frac{\partial}{\partial x}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} \phi dx = \frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} \frac{\partial \phi}{\partial x} dx = \overline{\frac{\partial \phi}{\partial x}}. \tag{11}$$

Moreover, the standard second-order approximation of the first-derivative at the face is exactly equal to the filtered derivative

$$\left.\frac{\partial \phi}{\partial x}\right|_{i+\frac{1}{2}} \approx \frac{\phi_{i+1}-\phi_i}{h_{i+\frac{1}{2}}} = \frac{1}{h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} \frac{\partial \phi}{\partial x} dx = \left.\overline{\frac{\partial \phi}{\partial x}}\right|_{i+\frac{1}{2}} \overset{Eq.(11)}{=} \left.\frac{\partial \overline{\phi}}{\partial x}\right|_{i+\frac{1}{2}}. \tag{12}$$

Remark 1 *This result suggests that the actual filter length when computing the face derivative is $h_{i+\frac{1}{2}}$, i.e. the distance between the adjacent nodes i and $i+1$ (see Figure 1).*

Finally, the diffusive term in a FVM framework is computed as follows

$$\begin{aligned}
\left.\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)\right|_i &\approx \frac{1}{h_i}\left(\Gamma\left.\frac{\partial \phi}{\partial x}\right|_{i+\frac{1}{2}} - \Gamma\left.\frac{\partial \phi}{\partial x}\right|_{i-\frac{1}{2}}\right) \\
&= \left.\overline{\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)}\right|_i \\
&\approx \frac{1}{h_i}\left(\Gamma_{i+\frac{1}{2}}\frac{\phi_{i+1}-\phi_i}{h_{i+\frac{1}{2}}} - \Gamma_{i-\frac{1}{2}}\frac{\phi_i-\phi_{i-1}}{h_{i-\frac{1}{2}}}\right) \\
&= \left.\overline{\frac{\partial}{\partial x}\left(\Gamma\overline{\frac{\partial \phi}{\partial x}}\right)}\right|_i .
\end{aligned} \tag{13}$$

In the case of non-constant diffusivity, $\Gamma(x)$, the evaluation of the previous expression can be viewed as filtering the Γ field, *i.e.*

$$\Gamma_{i+\frac{1}{2}} \approx \frac{\Gamma_i + \Gamma_{i+1}}{2} \approx \frac{1}{h_{i+\frac{1}{2}}}\int_{x_i}^{x_{i+1}} \Gamma(x) dx = \overline{\Gamma}_{i+\frac{1}{2}}, \tag{14}$$

while using the trapezoidal rule to approximate the integral. Altogether leads to

$$\begin{aligned}
&\left.\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)\right|_i \approx \\
&\frac{1}{h_i}\left(\frac{\Gamma_{i+1}+\Gamma_i}{2}\frac{\phi_{i+1}-\phi_i}{h_{i+\frac{1}{2}}} - \frac{\Gamma_i+\Gamma_{i-1}}{2}\frac{\phi_i-\phi_{i-1}}{h_{i-\frac{1}{2}}}\right) \\
&\qquad = \left.\overline{\frac{\partial}{\partial x}\left(\overline{\Gamma}\overline{\frac{\partial \phi}{\partial x}}\right)}^{FV}\right|_i ,
\end{aligned} \tag{15}$$

where $\overline{(\cdot)}^{FV}$ is the FVM filtering along the cell i whereas the other two filters are applied to the face quantities and their associated filter lengths are $h_{i+\frac{1}{2}}$ and $h_{i-\frac{1}{2}}$, respectively.

Remark 2 *This result suggests that two filtering operations are performed when computing the diffusive term: the calculation of the face derivative (see Eq.12) and the cell-to-face interpolation of the diffusivity (see Eq.14). Both filtering operators share the same filter length; namely, the distance between the nodes adjacent to the corresponding face.*

Consequently, this suggests that the SGS characteristic length used to compute the eddy viscosity, ν_t, at the face $i+\frac{1}{2}$ should be $h_{i+\frac{1}{2}}$ (see Figure 1). This concept forms the foundation of this work and can be easily extended to general meshes.

3 Rationale length scale

The subgrid characteristic length, δ, appears in a natural way when we consider the calculation of the ν_t at the cell faces. The flowchart shown in Figure 2 shows that it basically consists on firstly computing the $\hat{\boldsymbol{\nu}}_{t,c}$ at the cells without considering any characteristic length, then interpolating this quantity to the faces, $\hat{\boldsymbol{\nu}}_{t,s} = \Pi\hat{\boldsymbol{\nu}}_{t,c}$, where Π is a cell-to-face interpolation. Finally, this quantity at the face is re-scaled by the square of the cell distances contained in the diagonal matrix Δ_s (see [9] for details regarding the construction of Ω_c, Π and Δ_s). Therefore, from an implementation and conceptual point-of-view, the new approach δ_{rls} (rls stands for *r*ational *l*ength *s*cale) is completely different from all previous definitions of δ. Although the required code modifications are minimal (see flowchart in Figure 2), it may be of interest

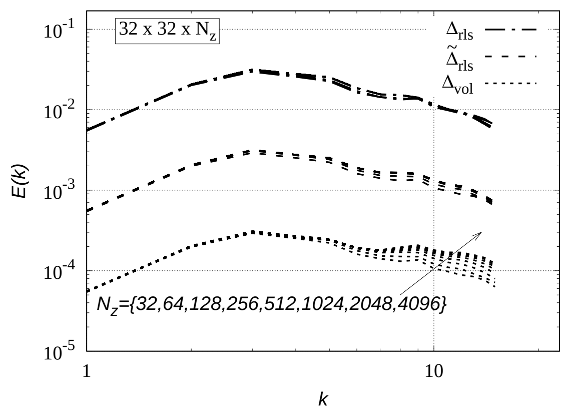

Figure 3: Energy spectra for decaying isotropic turbulence corresponding to the experiment of Comte-Bellot and Corrsin [10]. Results obtained with the new definitions δ_{rls} and $\tilde{\delta}_{\text{rls}}$ proposed in respectively Eqs.(15) and (18) are compared with the classical definition proposed by Deardorff given in Eq.(4). For clarity, the results obtained with $\tilde{\delta}_{\text{rls}}$ δ_{vol} are shifted down one and two decades, respectively.

to compute an equivalent length scale that provides the same dissipation. Namely, it can be shown that the local dissipation of the viscous term, with constant viscosity, is given by

$$\nu \mathsf{G} : \mathsf{G} = \nu tr(\mathsf{G}\mathsf{G}^T). \tag{16}$$

If we replace ν by ν_t, we can obtain a very accurate estimation of the local dissipation introduced by an eddy-viscosity model. Furthermore, in the new approach we also need to replace ν_t and G by $\hat{\nu}_t$ and G_δ, respectively, leading to

$$\hat{\nu}_t \mathsf{G}_\delta : \mathsf{G}_\delta = \hat{\nu}_t tr(\mathsf{G}_\delta \mathsf{G}_\delta^T). \tag{17}$$

Then, we can compute an equivalent filter length, $\tilde{\delta}_{\text{rls}}$, that leads to the same local dissipation, *i.e.*

$$\tilde{\delta}^2_{\text{rls}} \hat{\nu}_t \mathsf{G} : \mathsf{G} = \hat{\nu}_t \mathsf{G}_\delta : \mathsf{G}_\delta \qquad \Longrightarrow$$

$$\boxed{\tilde{\delta}_{\text{rls}} = \sqrt{\frac{\mathsf{G}_\delta : \mathsf{G}_\delta}{\mathsf{G} : \mathsf{G}}} = \sqrt{\frac{tr(\mathsf{G}_\delta \mathsf{G}_\delta^T)}{tr(\mathsf{G}\mathsf{G}^T)}}} \tag{18}$$

Notice that $P_{\mathsf{G}\mathsf{G}^T} = tr(\mathsf{G}\mathsf{G}^T)$ is the first invariant of the symmetric tensor $\mathsf{G}\mathsf{G}^T$.

4 Conclusions and preliminary results

This work proposes a novel definition of the subgrid characteristic length, δ, aimed at addressing the following research question: *can we establish a simple, robust, and easily implementable definition of δ for any type of grid that minimizes the impact of mesh anisotropies on the performance of SGS models?* In doing so, we firstly analyzed the entanglement between the numerical discretization and the filtering in LES. Preliminary results displayed in Figure 3 correspond to the classical experimental results obtained by Comte-Bellot and Corrsin [10]. LES results have been obtained using the Smagorinsky model, for a set of (artificially) stretched pancake-like meshes with $32 \times 32 \times N_z$ and $N_z = \{32, 64, 128, 256, 512, 1024, 2048, 4096\}$: the results obtained using the Deardorff definition, given in Eq.(4), tend to diverge as N_z increases. This is because the value of δ tends to vanish and, therefore, the subgrid-scale models switch off. In contrast, the newly proposed δ_{rls} and $\tilde{\delta}_{\text{rls}}$ avoid this issue, and the results rapidly converge for larger values of N_z. This suggests that they effectively reduce the influence of mesh anisotropies on SGS model performance. A detailed analysis and comparisons with other existing length scales will be presented in the final paper.

References

[1] F. X. Trias, D. Folch, A. Gorobets, and A. Oliva. Building proper invariants for eddy-viscosity subgrid-scale models. *Physics of Fluids*, 27(6):065103, 2015.

[2] J. W. Deardorff. Numerical study of three-dimensional turbulent channel flow at large Reynolds numbers. *Journal of Fluid Mechanics*, 41:453–480, 1970.

[3] A. Scotti, C. Meneveau, and D. K. Lilly. Generalized Smagorinsky model for anisotropic grids. *Physics of Fluids A*, 5(9):2306–2308, 1993.

[4] M. L. Shur, P. R. Spalart, M. K. Strelets, and A. K. Travin. An Enhanced Version of DES with Rapid Transition from RANS to LES in Separated Flows. *Flow, Turbulence and Combustion*, 95:709–737, 2015.

[5] C. Mockett, M. Fuchs, A. Garbaruk, M. Shur, P. Spalart, M. Strelets, F. Thiele, and A. Travin. Two Non-zonal Approaches to Accelerate RANS to LES Transition of Free Shear Layers in DES. In Sharath Girimaji, Werner Haase, Shia-Hui Peng, and Dieter Schwamborn, editors, *Progress in Hybrid RANS-LES Modelling*, volume 130 of *Notes on Numerical Fluid Mechanics and Multidisciplinary Design*, pages 187–201. Springer International Publishing, 2015.

[6] P. R. Spalart, W. H. Jou, M. Strelets, and S. R. Allmaras. Comments on the feasibility of LES for wings, and on a hybrid RANS/LES approach. In *C. Liu & Z. Liu (eds.) Advances in DES/LES*. Greyden Press, Columbus, OH, 1997.

[7] N. Chauvet, S. Deck, and L. Jacquin. Zonal detached eddy simulation of a controlled propulsive jet. *AIAA Journal*, 45(10):2458–2473, 2007.

[8] F. X. Trias, A. Gorobets, M. H. Silvis, R. W. C. P. Verstappen, and A. Oliva. A new subgrid characteristic length for turbulence simulations on anisotropic grids. *Physics of Fluids*, 26:115109, 2017.

[9] F. X. Trias, X. Álvarez-Farré, À. Alsalti-Baldellou, A. Gorobets, and A. Oliva. An efficient eigenvalue bounding method: CFL condition revisited. *Computer Physics Communications*, 305:109351, 2024.

[10] G. Comte-Bellot and S. Corrsin. Simple Eulerian time correlation of full- and narrow-band velocity signals in grid-generated, isotropic turbulence. *Journal of Fluid Mechanics*, 48:273–337, 1971.