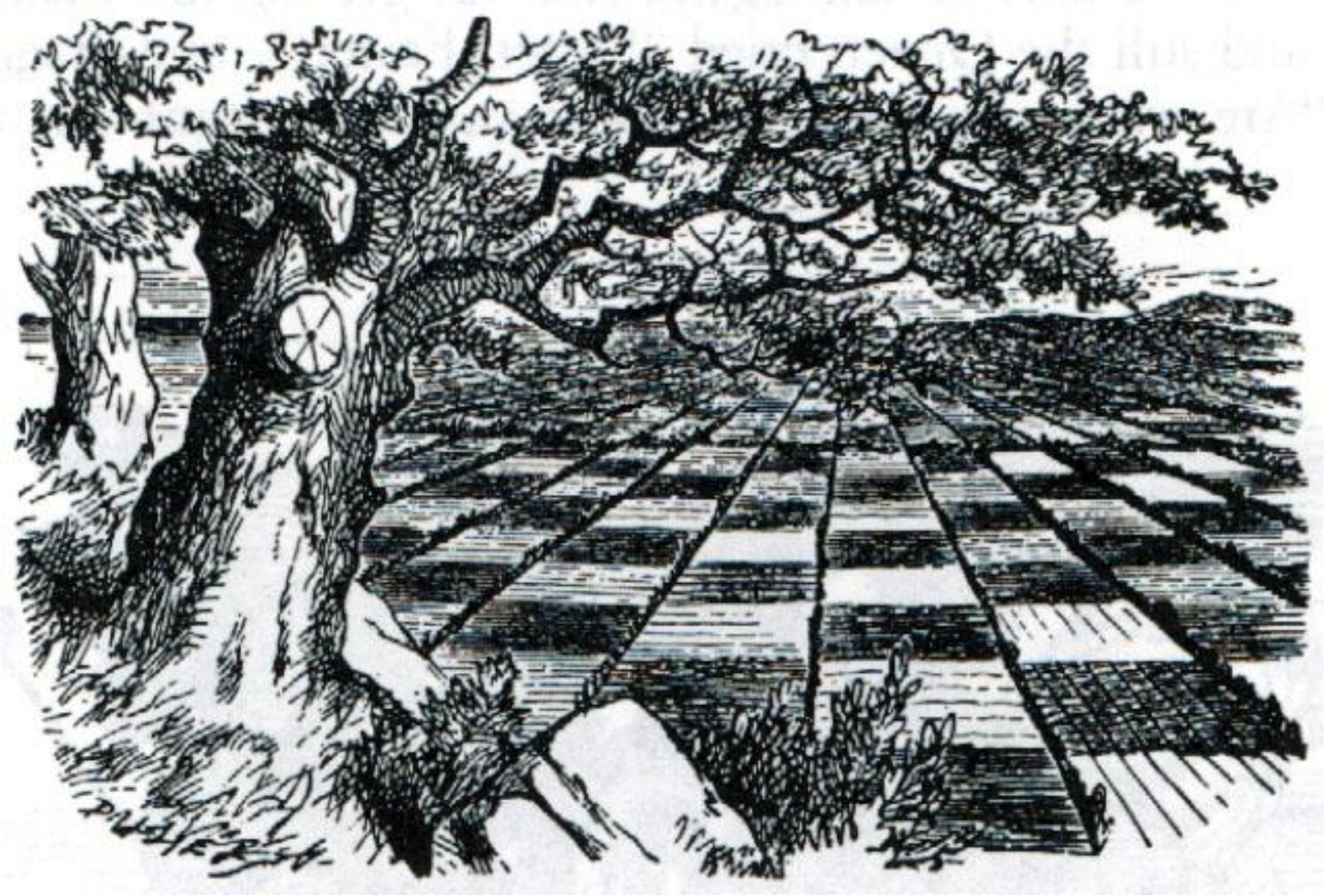
AN UNCONDITIONALLY STABLE, ENERGY PRESERVING METHOD FOR MAGNETOHYDRODYNAMICS

J.A. Hopman, J. Rigola, F.X. Trias

D. Santos



UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

ERCOFTAC European Research Community On Flow, Turbulence And Combustion

Motivation

MagnetoHydroDynamic (MHD) flows for Nuclear Fusion

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-High Hartmann number, $Ha = LB_0\sqrt{\frac{\sigma_0}{\rho_0 \nu}}$

-Low magnetic Reynolds number, $Re_m = \sigma \mu L u_0 \ll 1$

Challenges

Complex geometries

Balancing Lorentz force & pressure drop

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→ Collocated grids

Balancing Lorentz force & pressure drop

Challenges

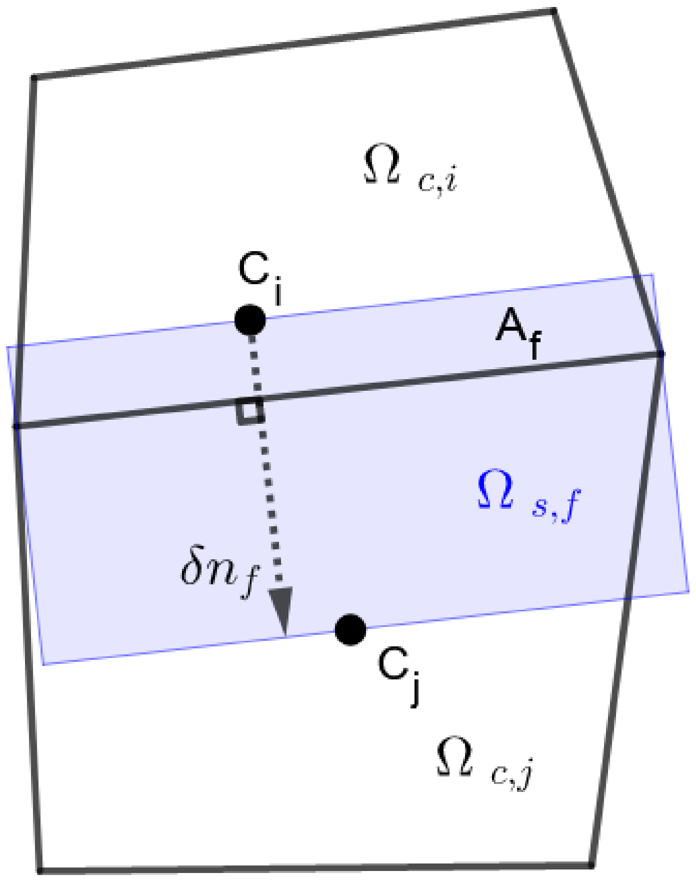
Complex geometries

→ Collocated grids

Balancing Lorentz force & pressure drop

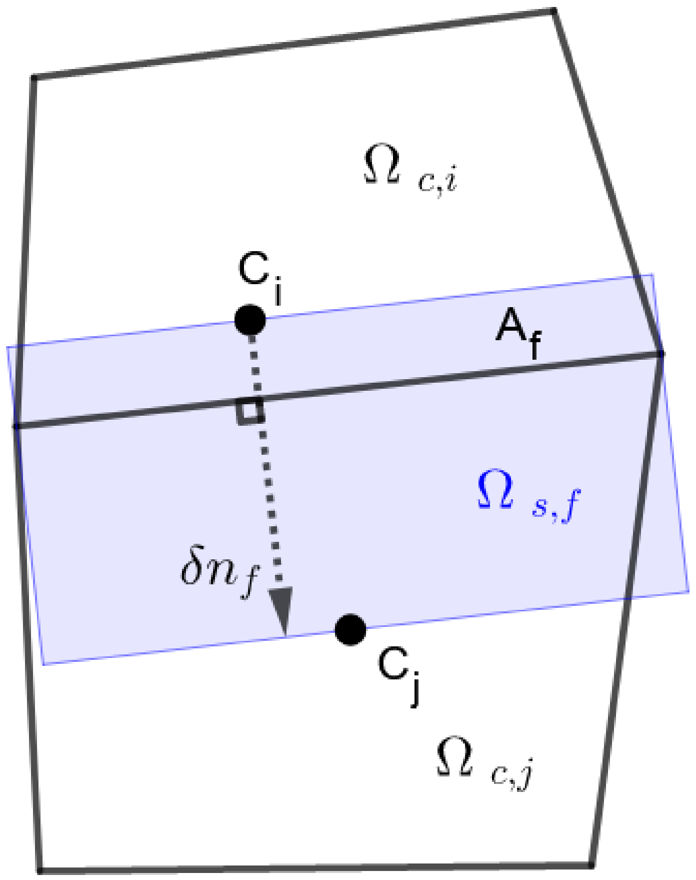
→ Symmetry-preserving method

Symmetry-Preserving method



Symmetry-Preserving method

1. Projected gradient distances



Symmetry-Preserving method

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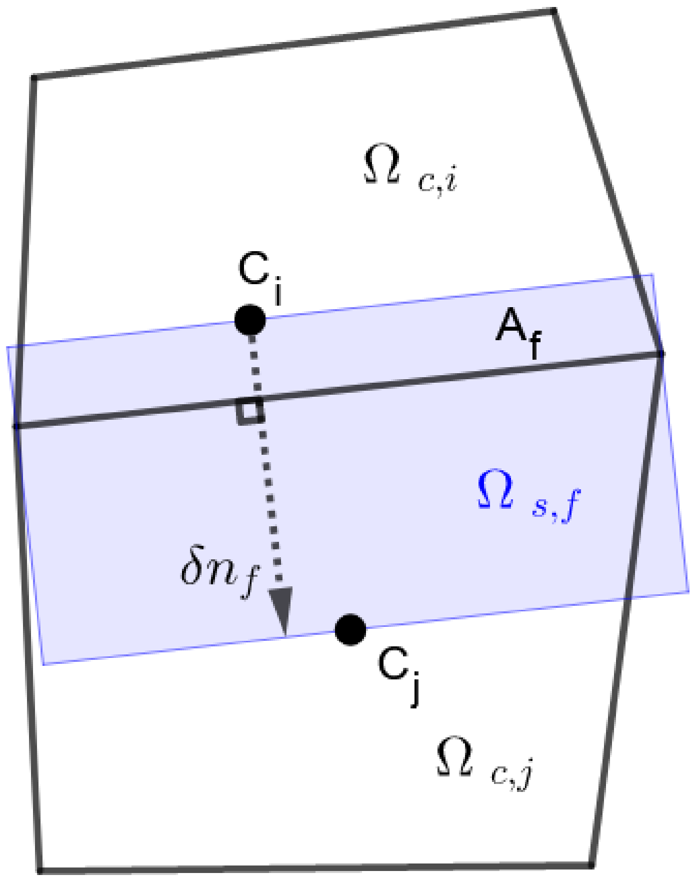
2. Consistent Div, Grad, Lap



Symmetry-Preserving method

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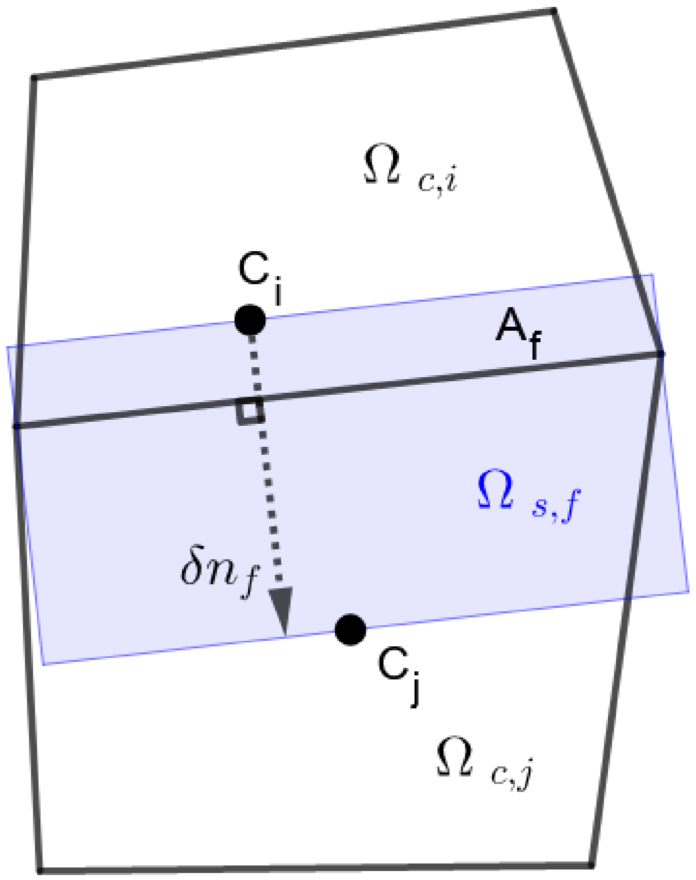
3. Midpoint interpolation in $C(\mathbf{u}_s)$



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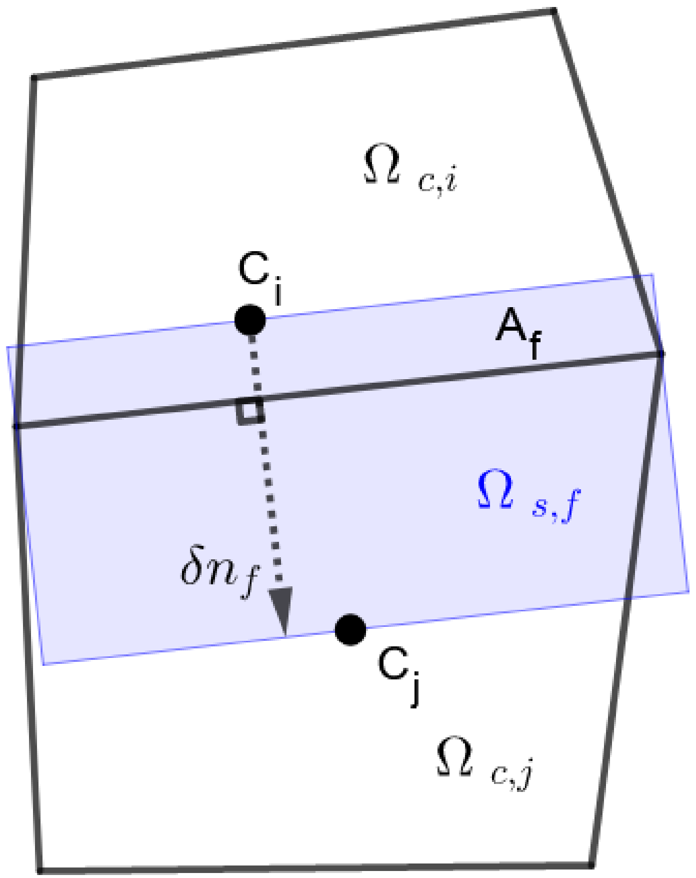
4. Volumetric interpolation



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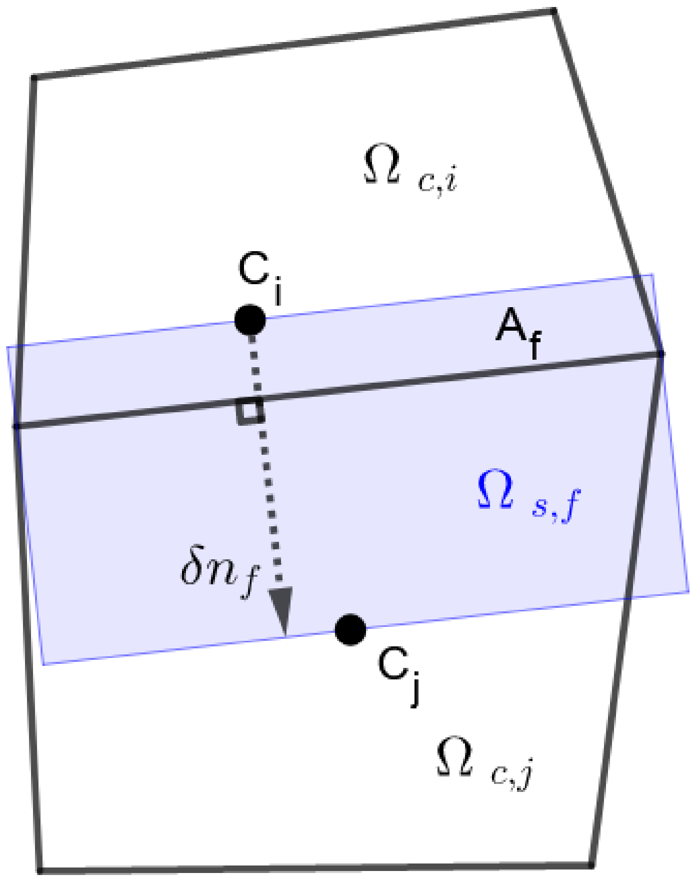
   - Flux term of Poisson equation



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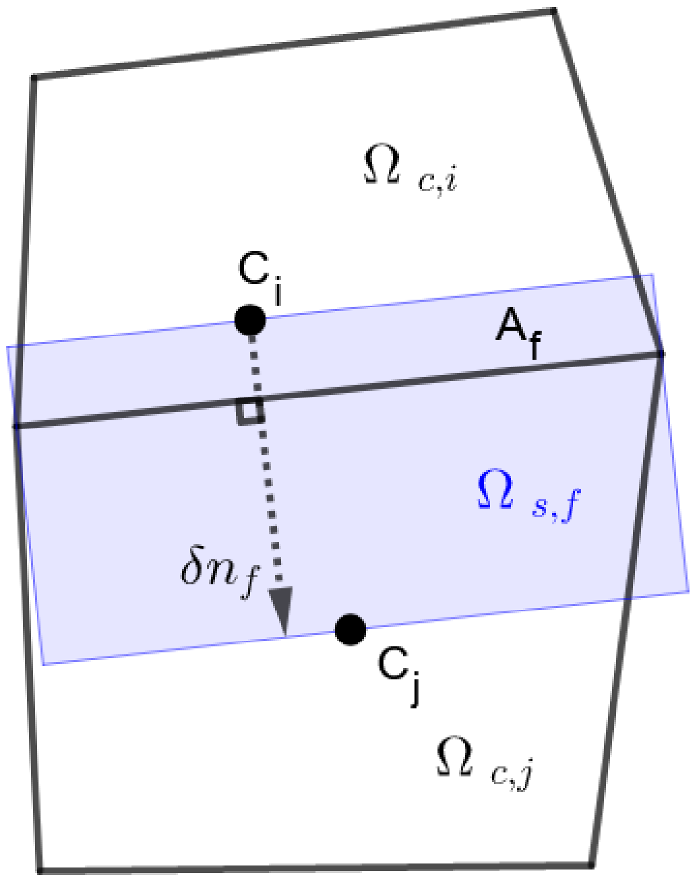
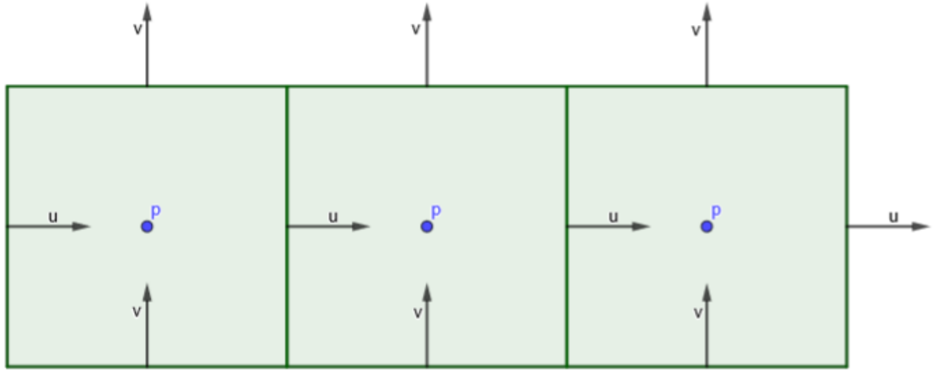
   - Flux term of Poisson equation
   - Correction term after Poisson



Checkerboarding



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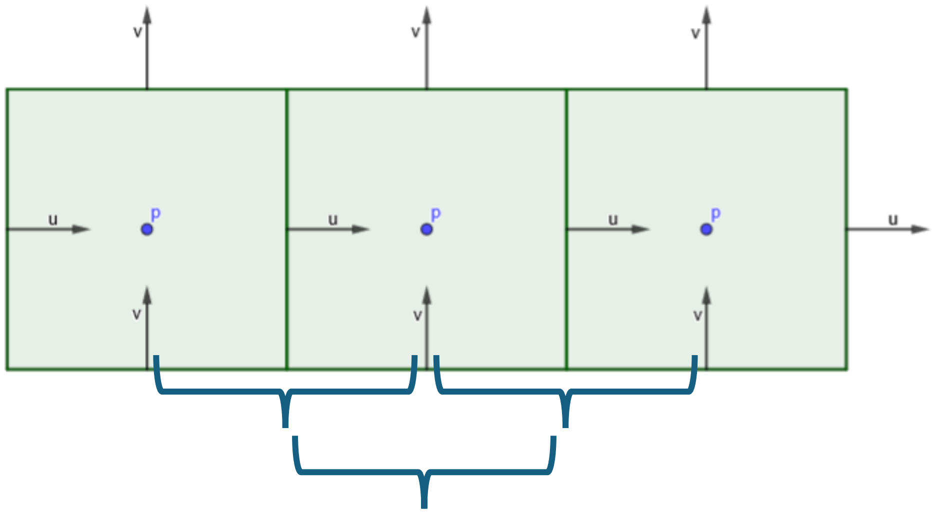
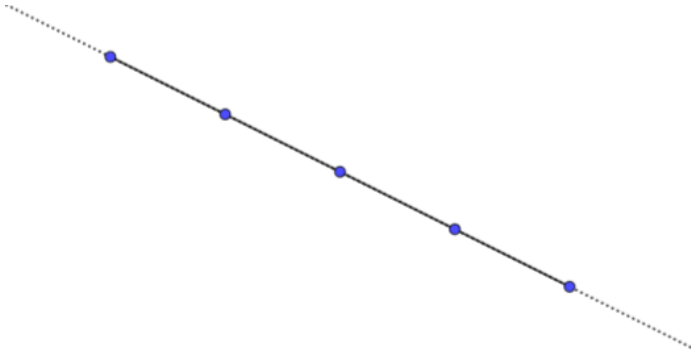


$L = MG$ compact stencil



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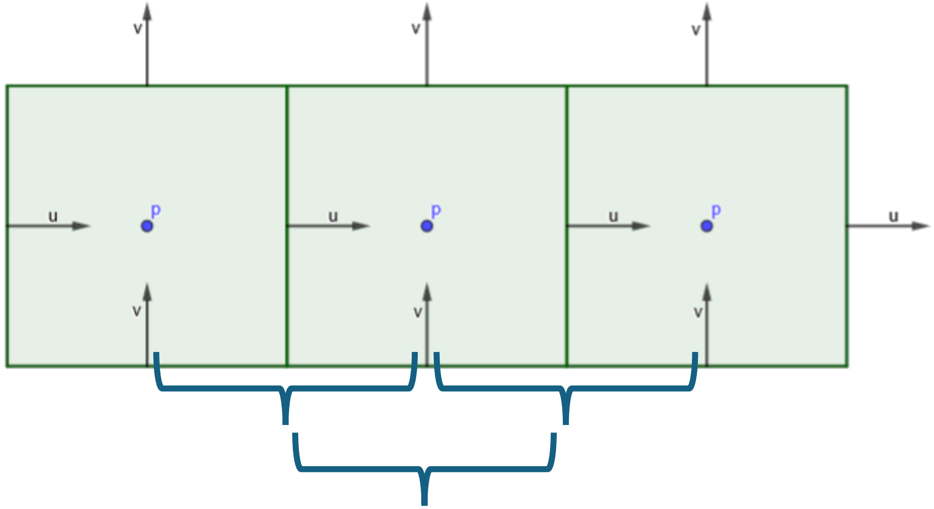

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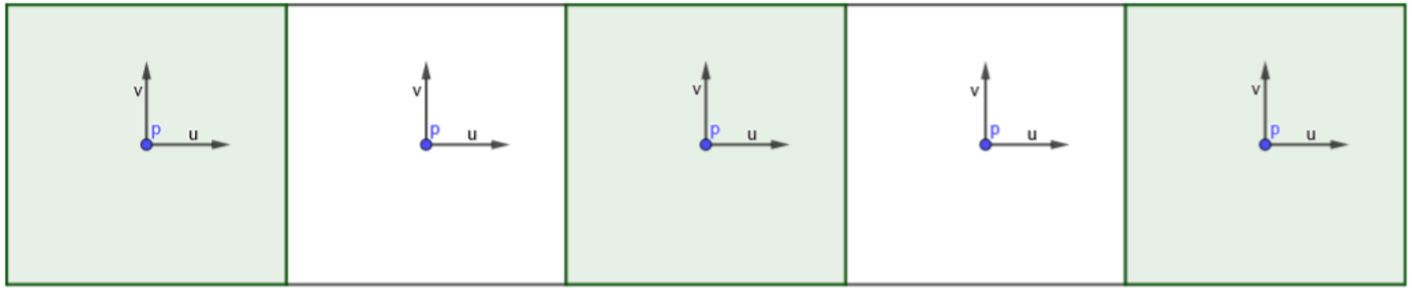

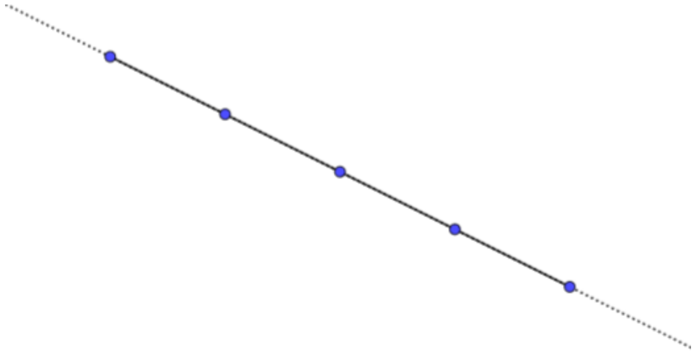

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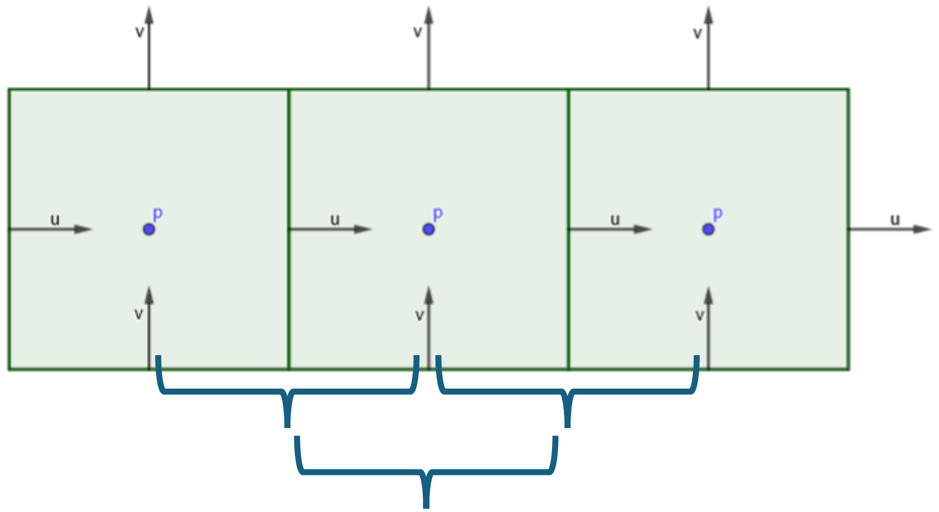

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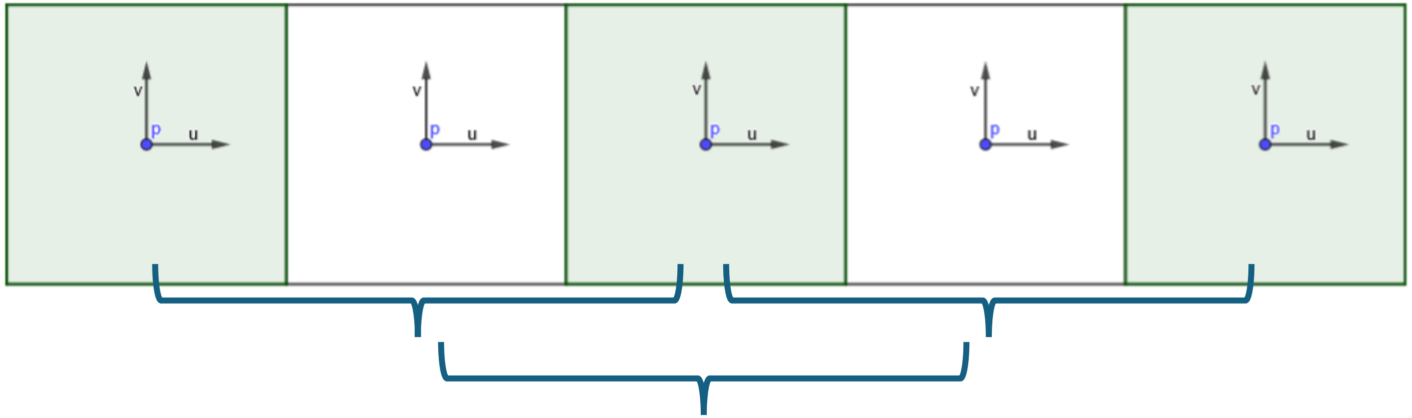
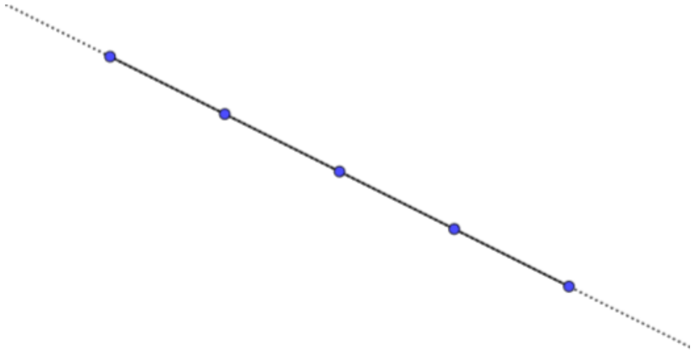
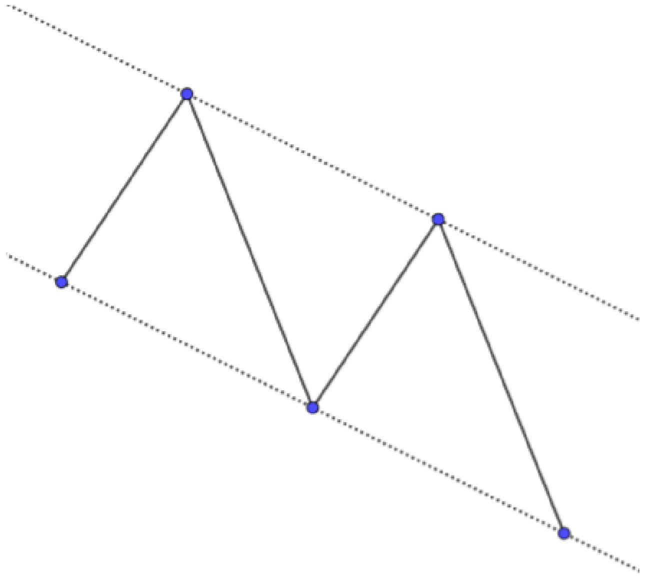


$L_c = M_c G_c$ wide stencil → decoupling





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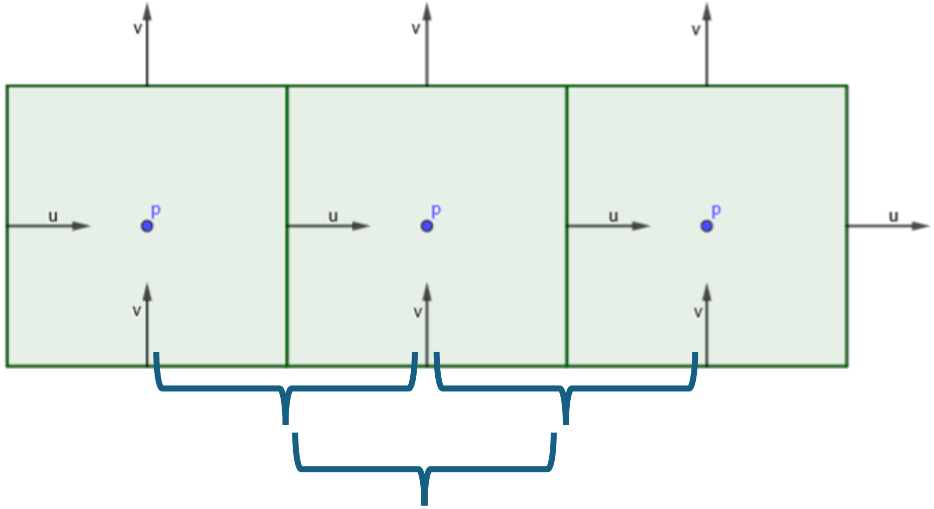

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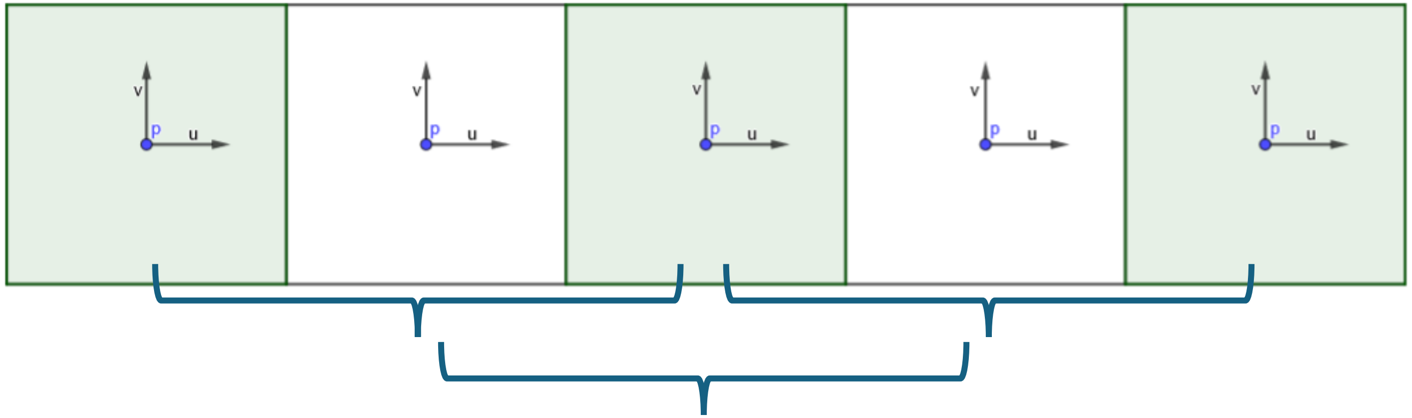

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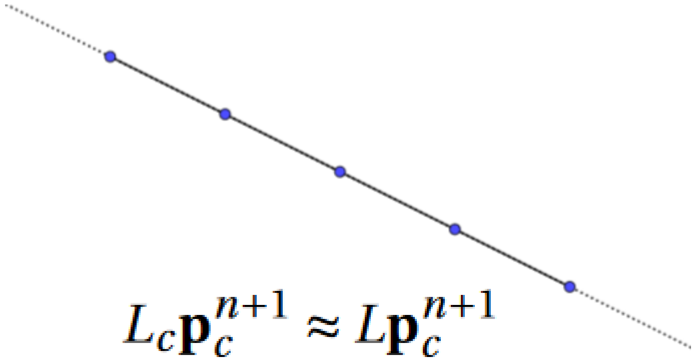
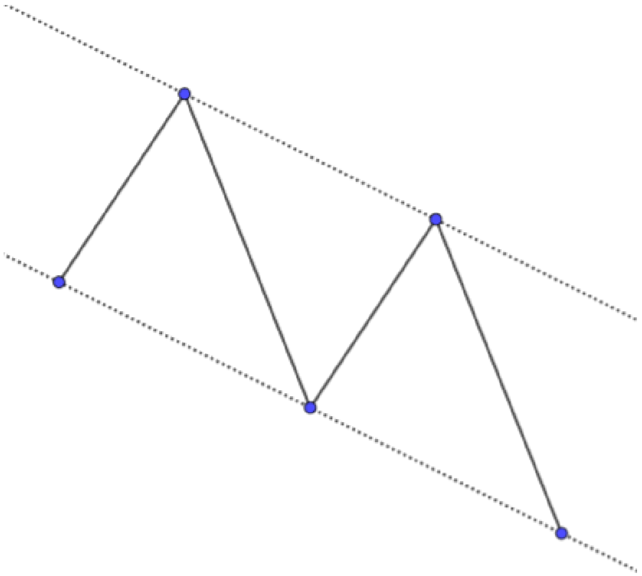


$L_c \mathbf{p}_c^{n+1} \approx L \mathbf{p}_c^{n+1}$

→ pressure error



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Lowering pressure error ~ $(L - L_c)\,\tilde{\mathbf{p}}'_c$

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Lowering pressure error ~ $(L - L_c)\,\tilde{\mathbf{p}}_c'$

Larger part on L_c → More prone to checkerboarding

Induction-less approximation

Formulation of second Poisson equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla (p/\rho) + (\mathbf{J} \times \mathbf{B})/\rho, \qquad \nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{J} = \sigma (-\nabla \phi + \mathbf{u} \times \mathbf{B}), \qquad \nabla \cdot \mathbf{J} = 0.$$

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Quantifying checkerboarding

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Rearranging to:

$$C_{cb} = 1 - \frac{\mathbf{p}_c^T L_c \mathbf{p}_c}{\mathbf{p}_c^T L \mathbf{p}_c}$$

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Balance checkerboarding and accuracy

General predictor coefficient:

$$\theta_a = \frac{a_c^T L_c a_c}{a_c^T L a_c}$$

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Low Cb

Use more predictor

Balance checkerboarding and accuracy

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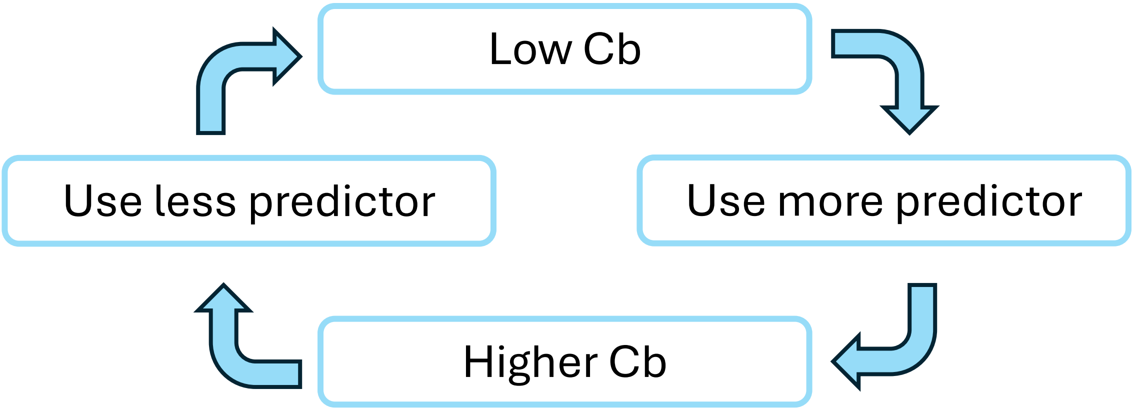
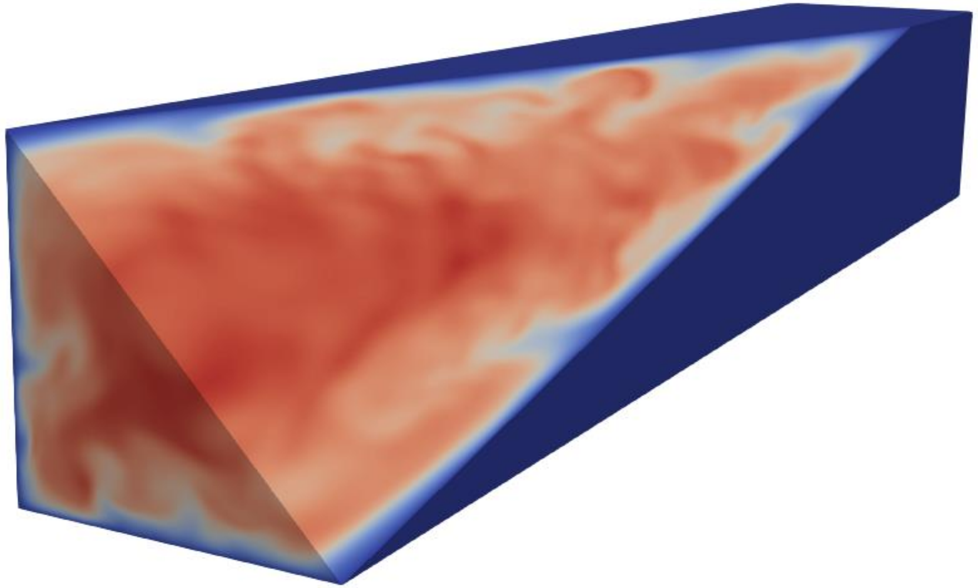
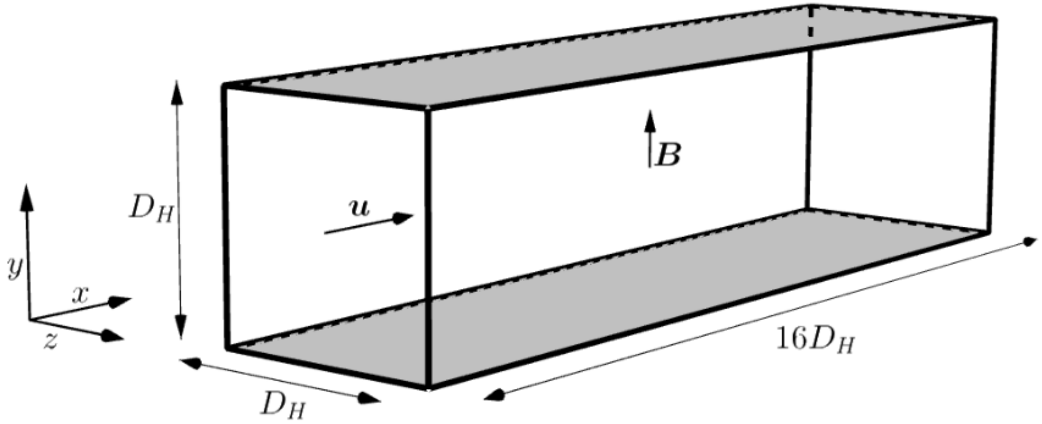


Results – Turbulent duct flow





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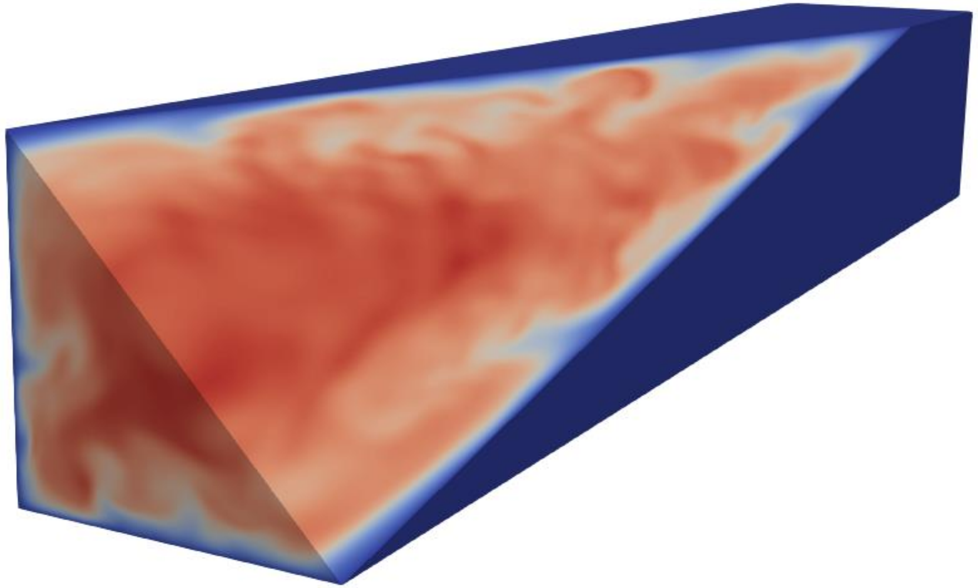
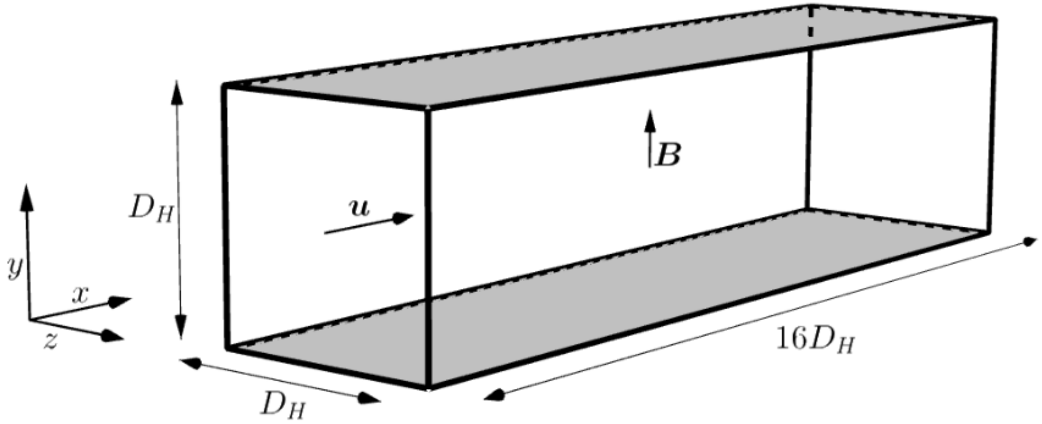


Hydrodynamic



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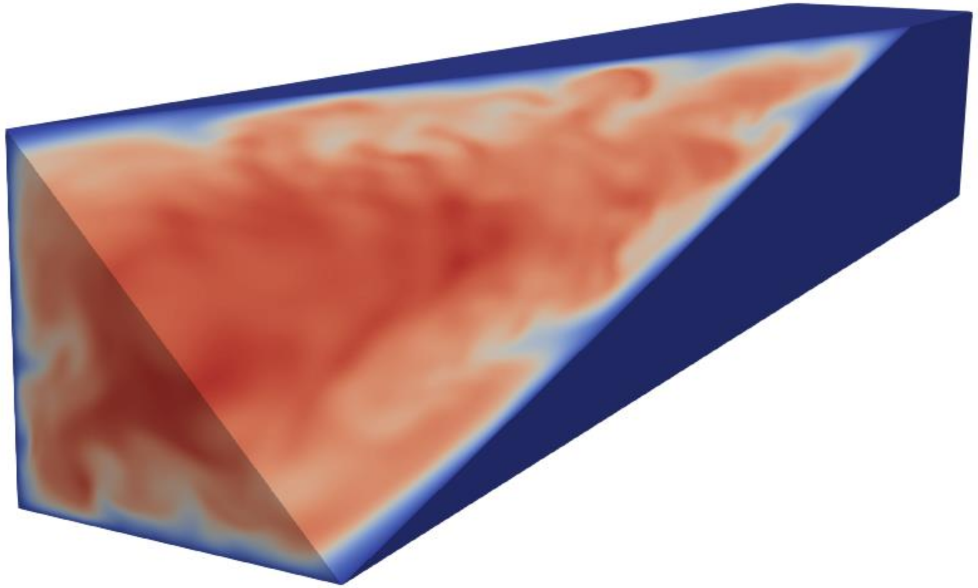

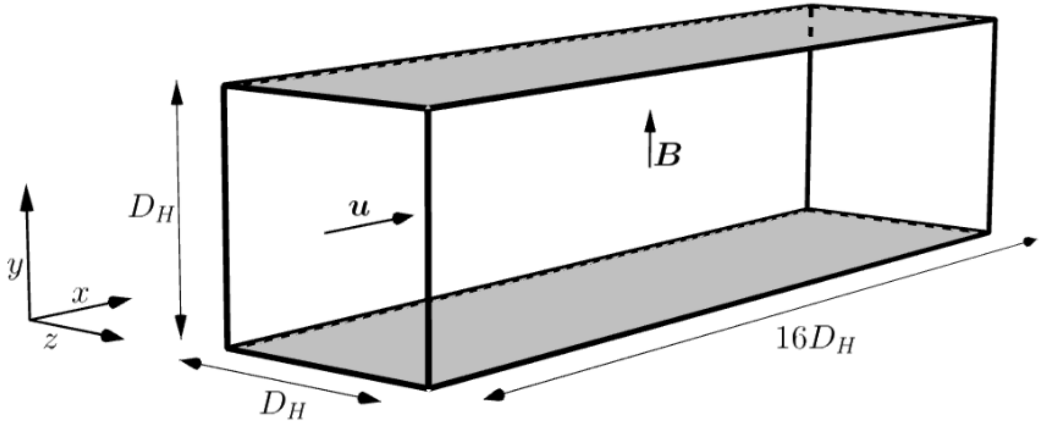

Hydrodynamic
MHD

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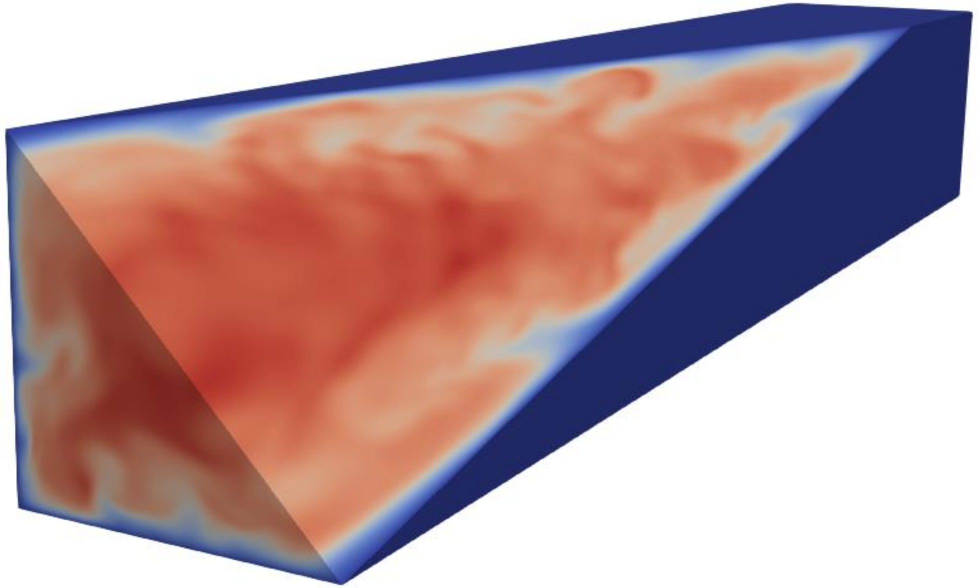

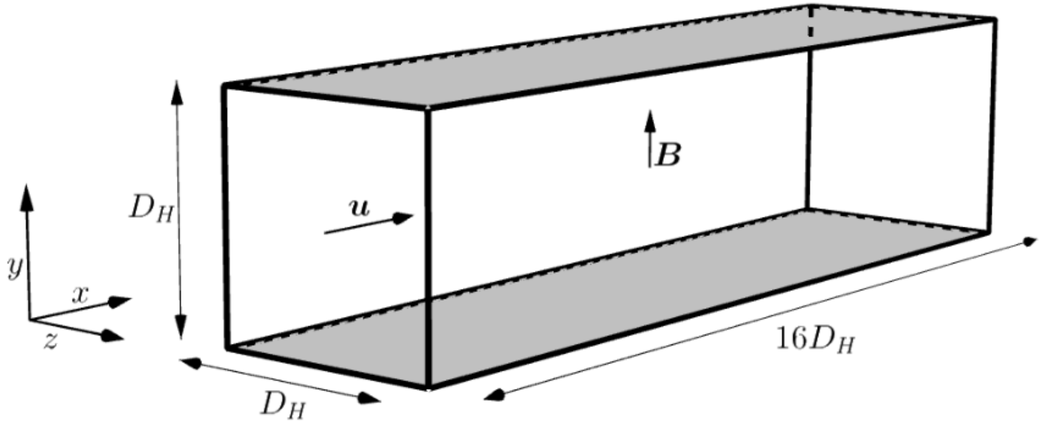

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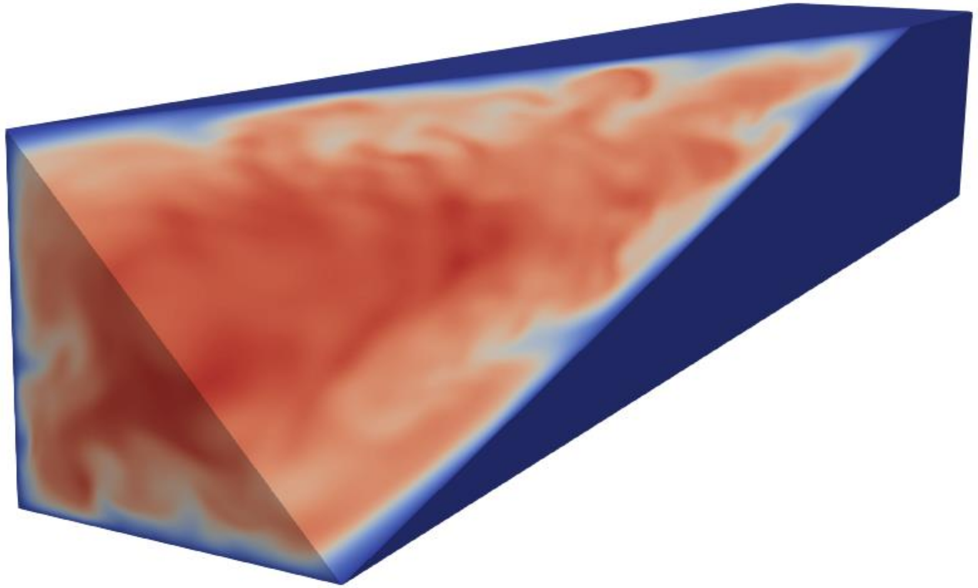

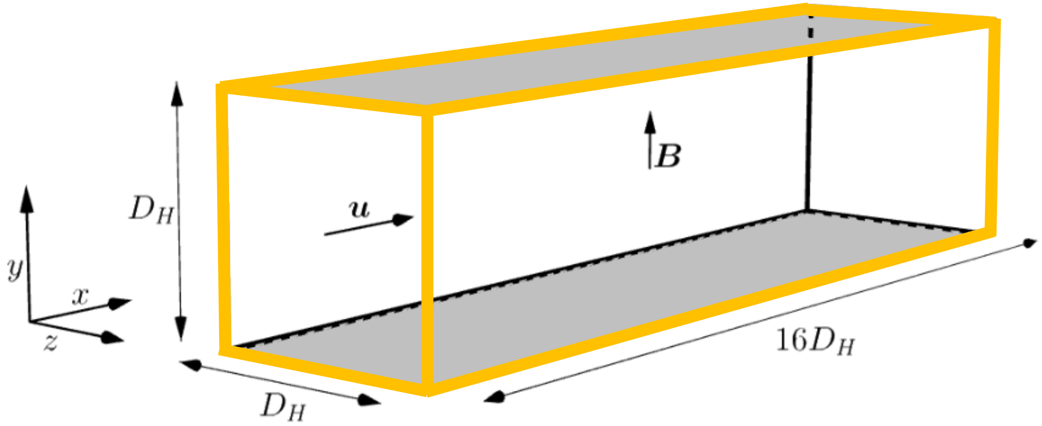

Hydrodynamic

MHD $\left\{ \begin{array}{l} \text{Insulated (Shercliff)} \\ \text{Conductive} \end{array} \right.$

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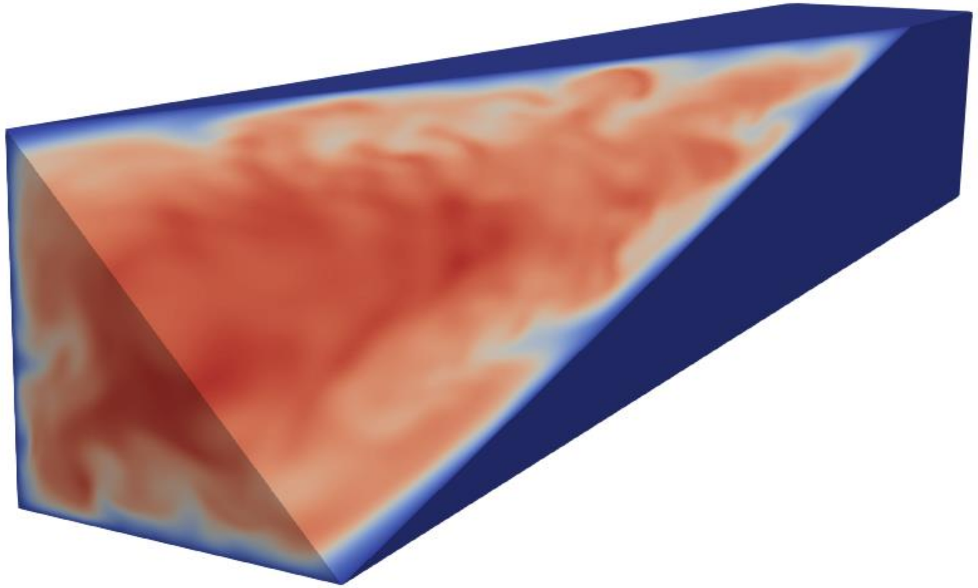

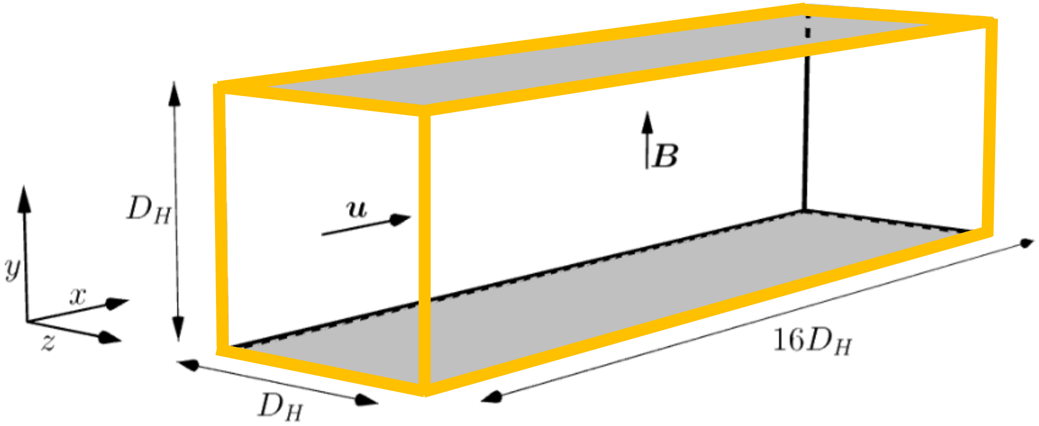

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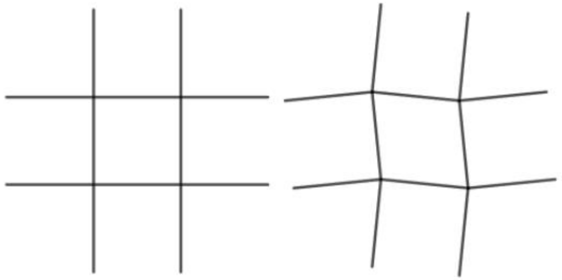

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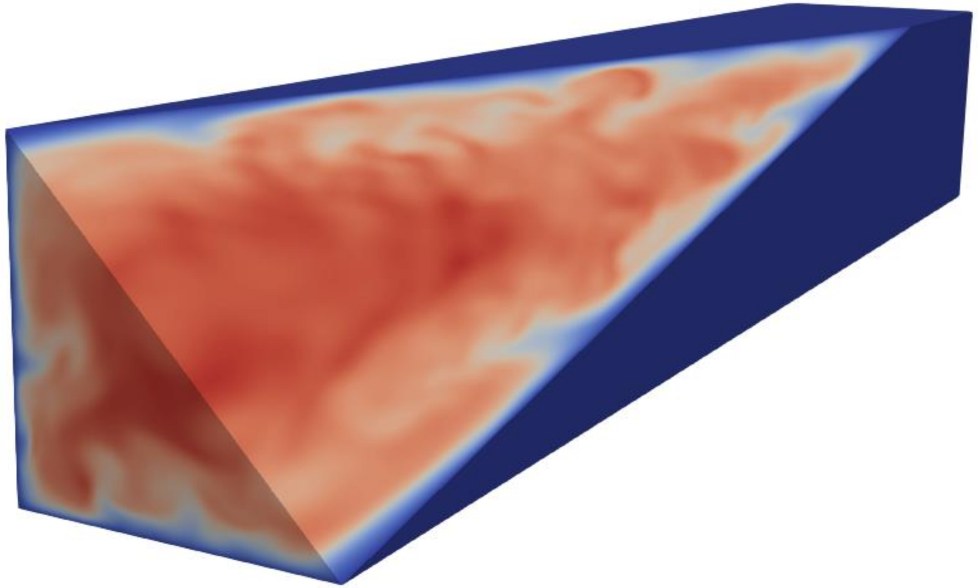

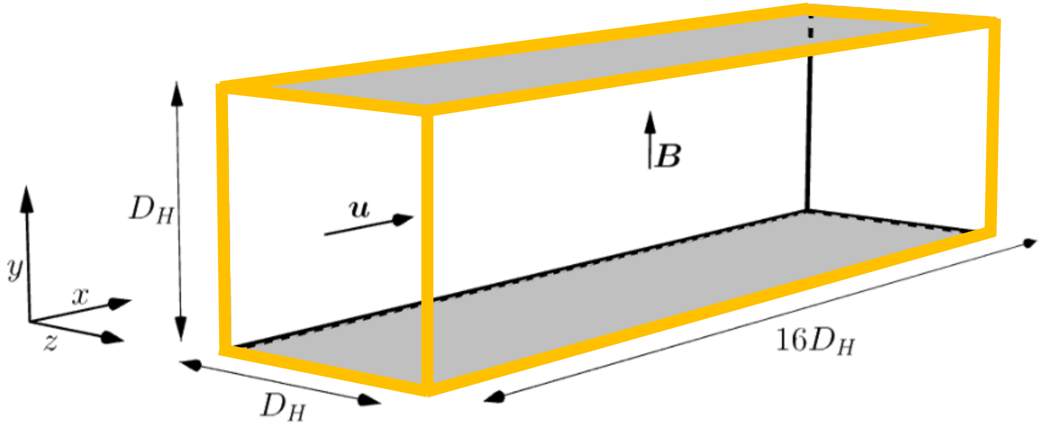

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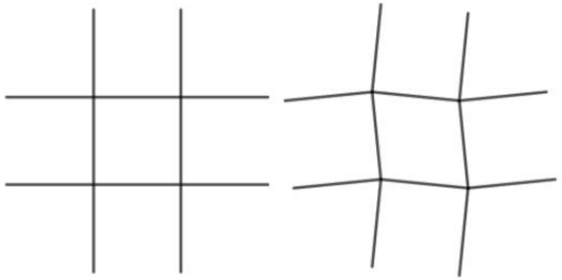
Mesh	L_x	N_x	$N_{\{y,z\}}$	$N_{tot}\,(\cdot 10^6)$	Δx^+	$\Delta\{y,z\}^+_B$	$\Delta\{y,z\}^+_w$
H_{coarse}	$2\pi D_H$	160	64	0.66	11.8	9.1	0.9
H_{fine}	$2\pi D_H$	160	128	2.62	11.8	4.6	0.4
M_{coarse}	$16D_H$	160	64	0.66	36.0	10.9	1.1
M_{fine}	$16D_H$	320	128	5.24	18.0	5.5	0.5



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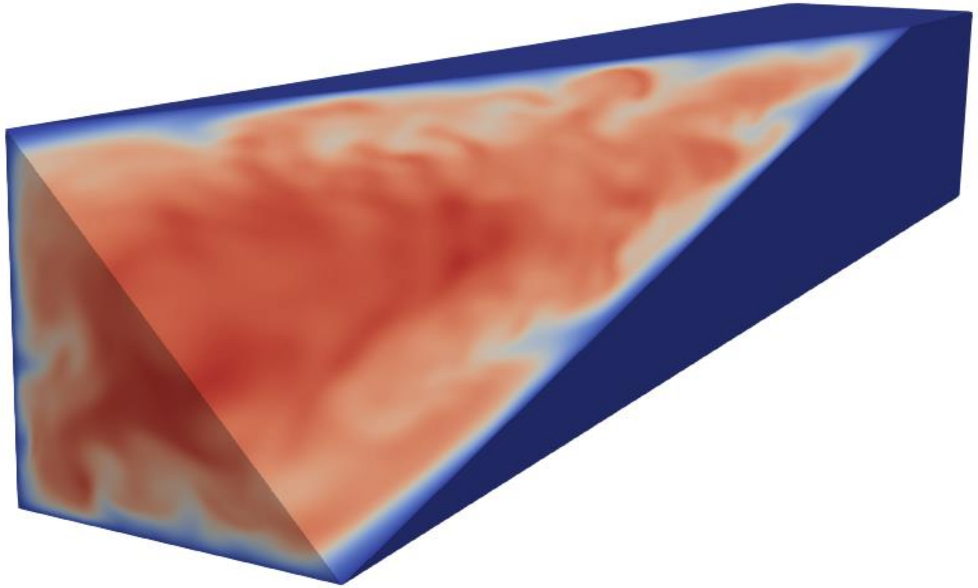
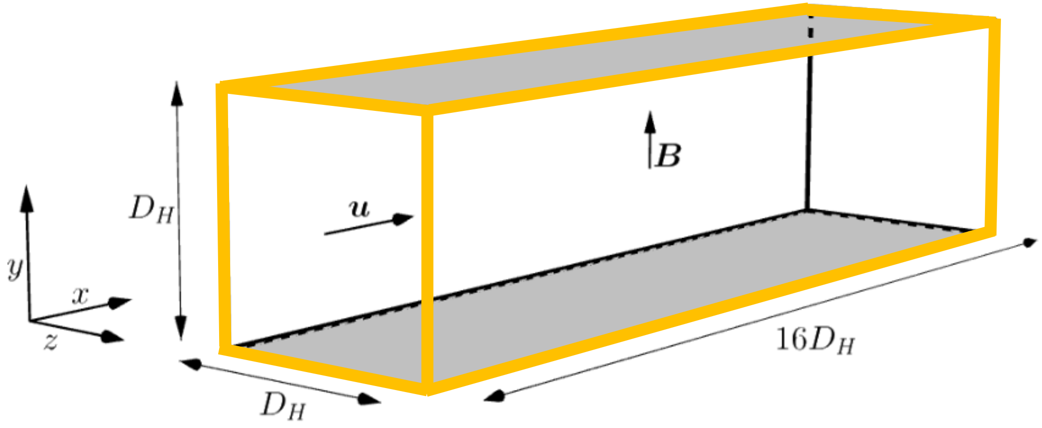
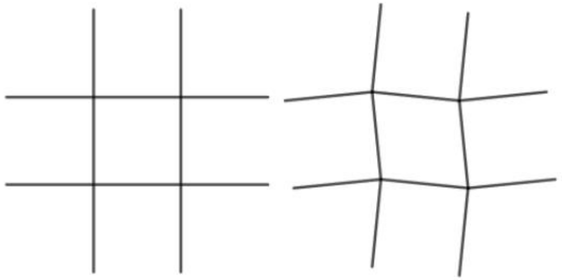




Hydrodynamic
MHD

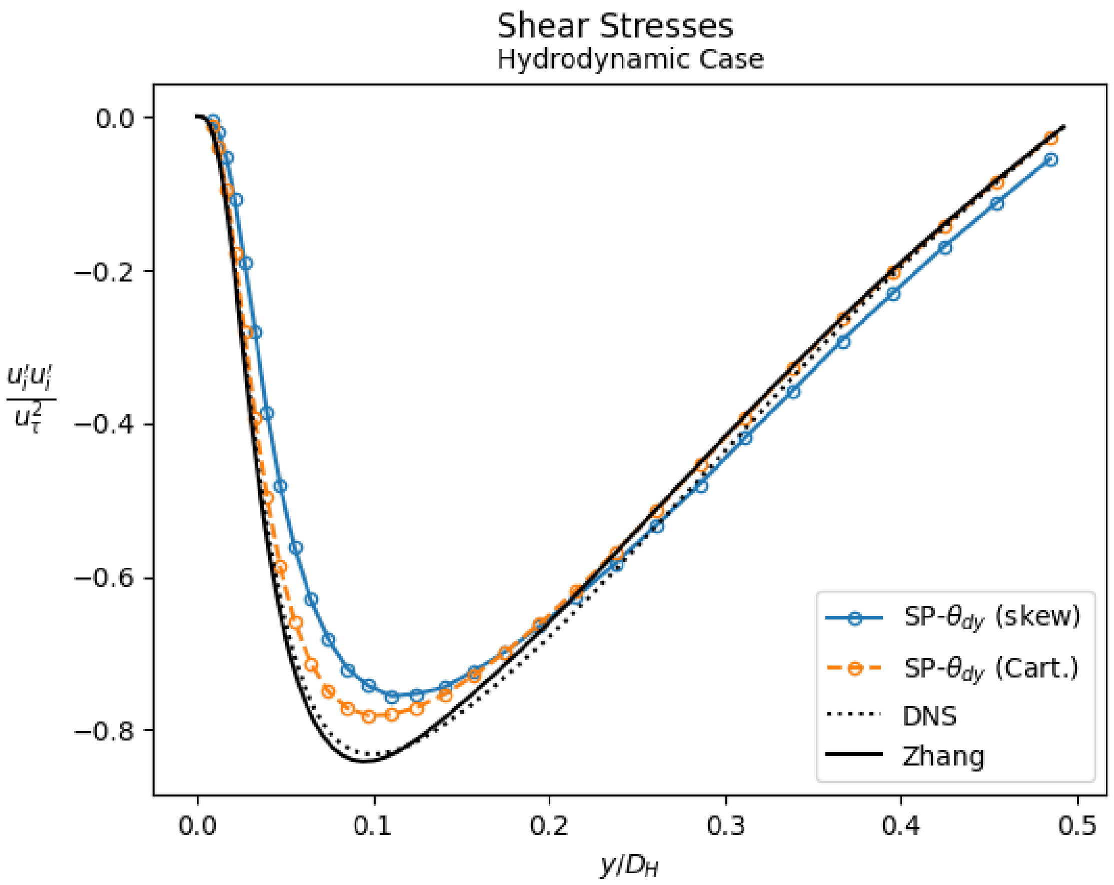
$\left\{ \begin{array}{l} \text{Insulated (Shercliff)} \\ \text{Conductive} \end{array} \right.$

Mesh	L_x	N_x	$N_{\{y,z\}}$	$N_{tot}\,(\cdot 10^6)$	Δx^+	$\Delta\{y,z\}_B^+$	$\Delta\{y,z\}_w^+$
H_{coarse}	$2\pi D_H$	160	64	0.66	11.8	9.1	0.9
H_{fine}	$2\pi D_H$	160	128	2.62	11.8	4.6	0.4
M_{coarse}	$16 D_H$	160	64	0.66	36.0	10.9	1.1
M_{fine}	$16 D_H$	320	128	5.24	18.0	5.5	0.5

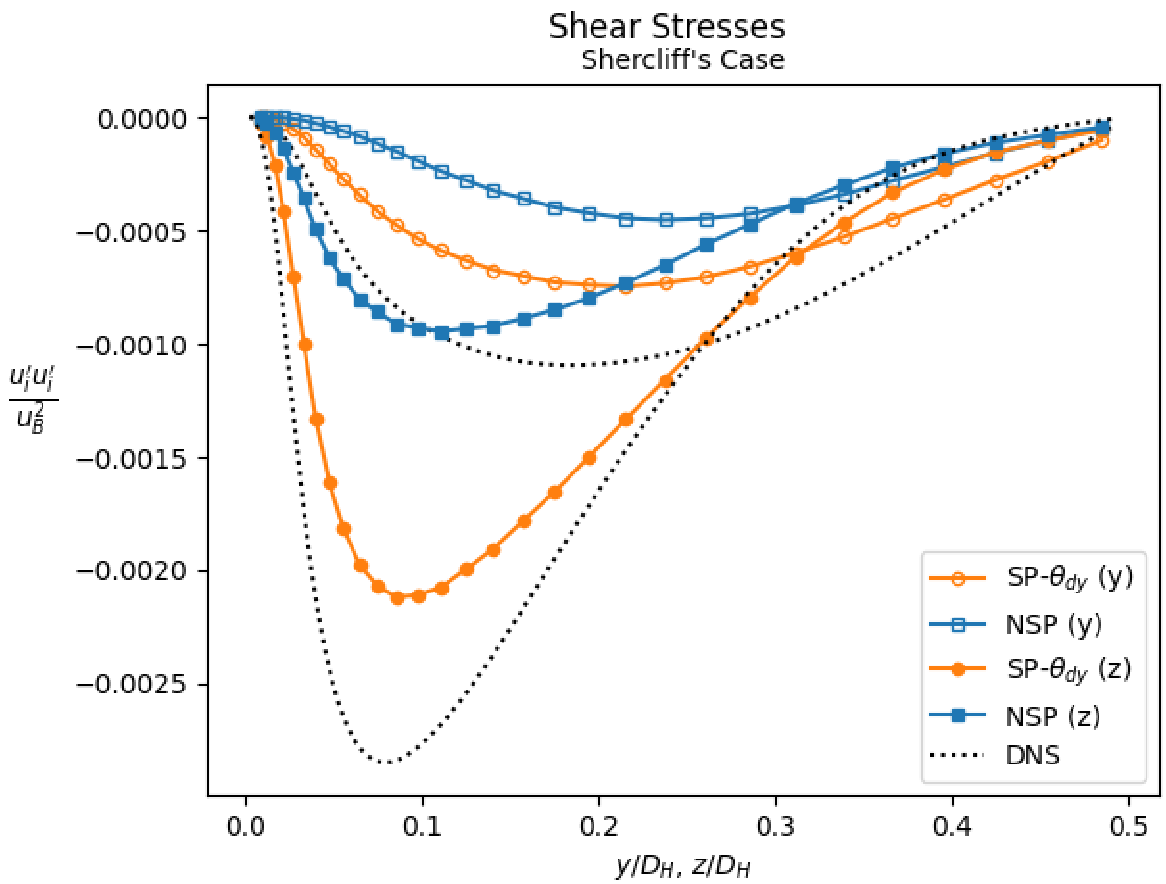


$$\underbrace{\frac{1}{\rho}\overline{u_i'\left(\mathbf{J}\times\mathbf{B}\right)_i'}}_{\text{Net MHD work}} = \underbrace{-\frac{\sigma}{\rho}\overline{u_i'\left(\left(\nabla\phi\right)\times\mathbf{B}\right)_i'}}_{M_k^\phi} + \underbrace{\frac{\sigma}{\rho}\overline{u_i'\left(\mathbf{u}\times\mathbf{B}\times\mathbf{B}\right)_i'}}_{M_k^u}$$

Results - Turbulent hydrodynamic duct flow



Results - Turbulent MHD duct flow



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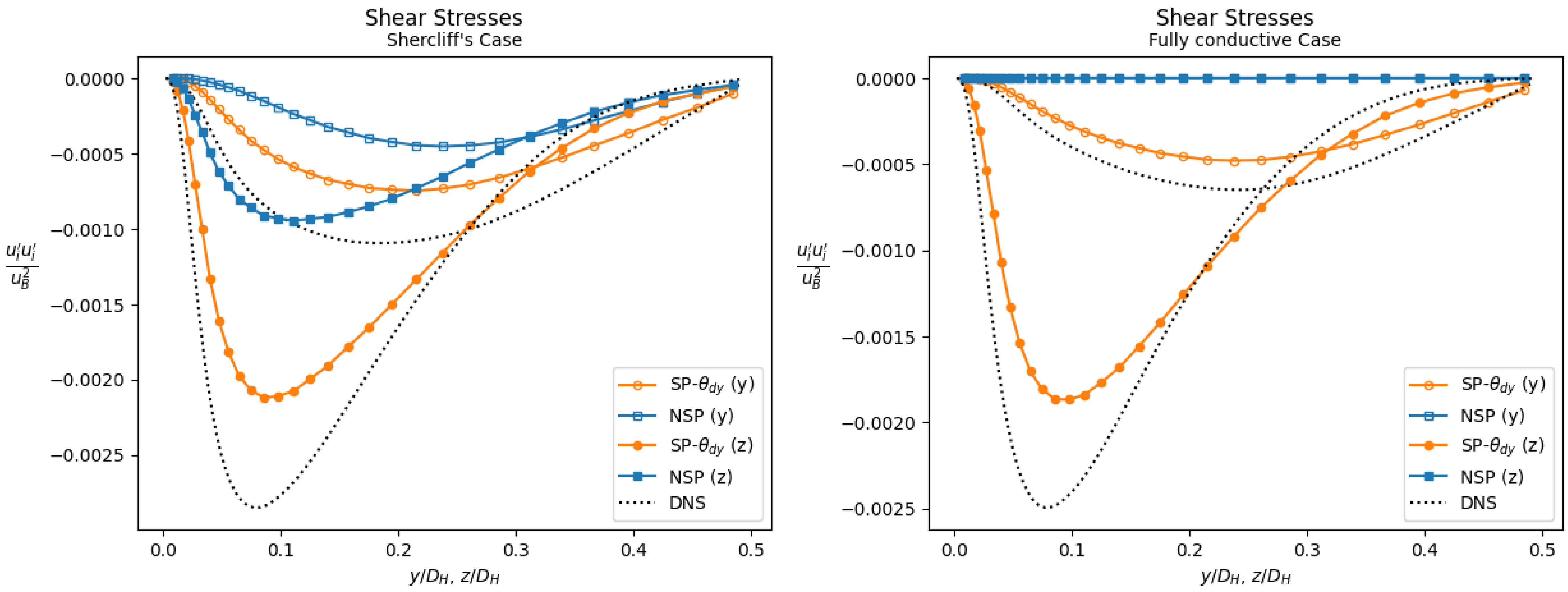

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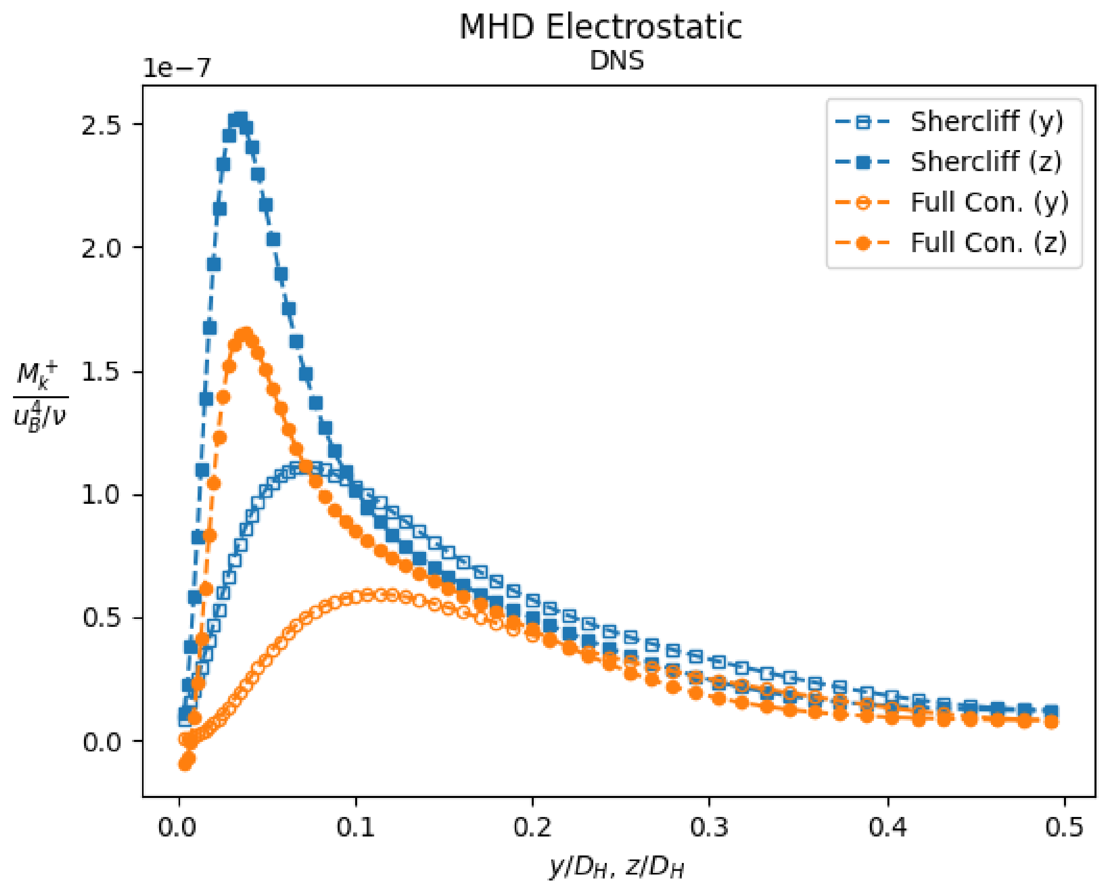

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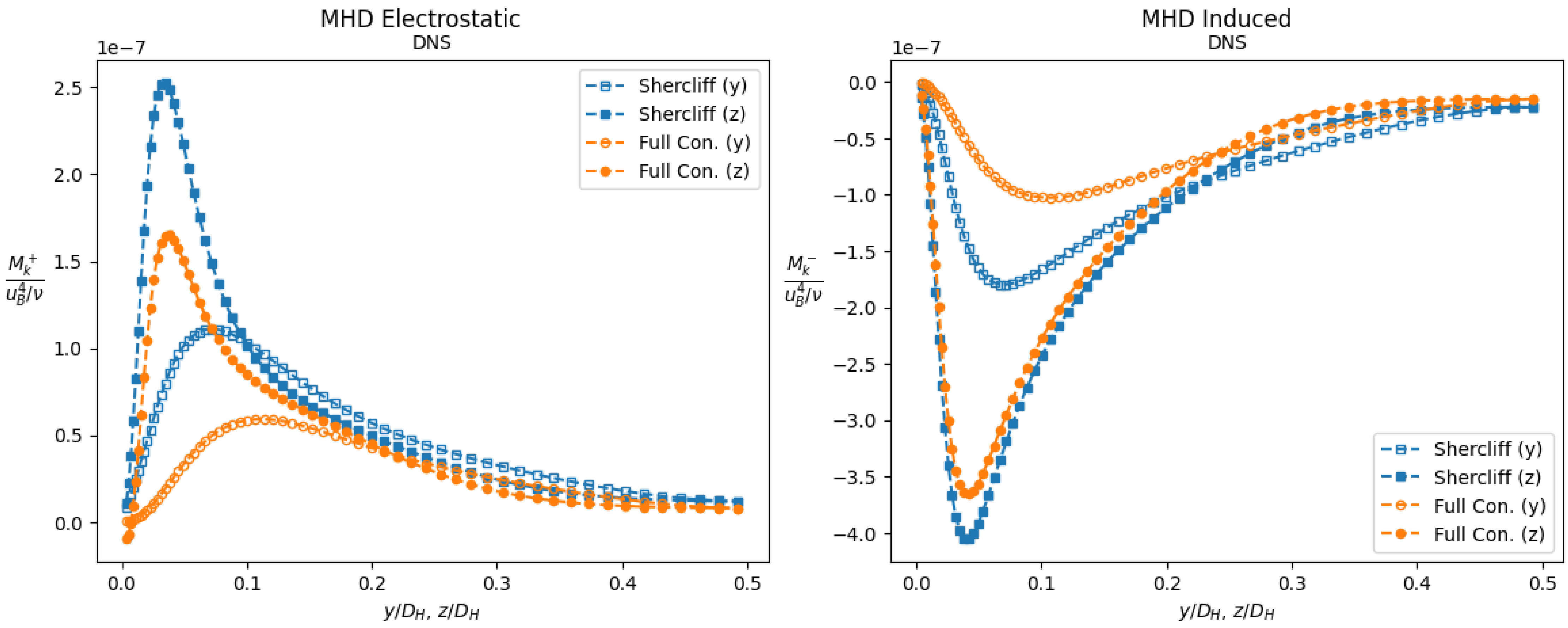

Results - Turbulent MHD duct flow

			SP-θ_{dy}			NSP		
Case	Mesh	Cells	p_{cb}	$\phi_{cb}\,(\cdot 10^{-3})$	$Re_{\{\tau/B\}}$	p_{cb}	$\phi_{cb}\,(\cdot 10^{-3})$	$Re_{\{\tau/B\}}$
Hydrodyn.	H_{fine}	Cart.	0.14	0	$4314[B]$	-	-	-
"	H_{coarse}	Cart.	0.32	0	$4536[B]$	0.37	0	$4389[B]$
"	H_{coarse}	skew.	0.49	0	$4787[B]$	0.80	0	$4826[B]$
Shercliff	M_{fine}	Cart.	0.13	0.17	$368.4[\tau]$	-	-	-
"	M_{coarse}	skew	0.50	0.64	$351.7[\tau]$	0.89	0.61	$328.8[\tau]$
Full Con.	M_{fine}	Cart.	0.12	2.75	$364.3[\tau]$	-	-	-
"	M_{coarse}	skew	0.49	8.33	$351.2[\tau]$	0.87	3.62	$317.9[\tau]$

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