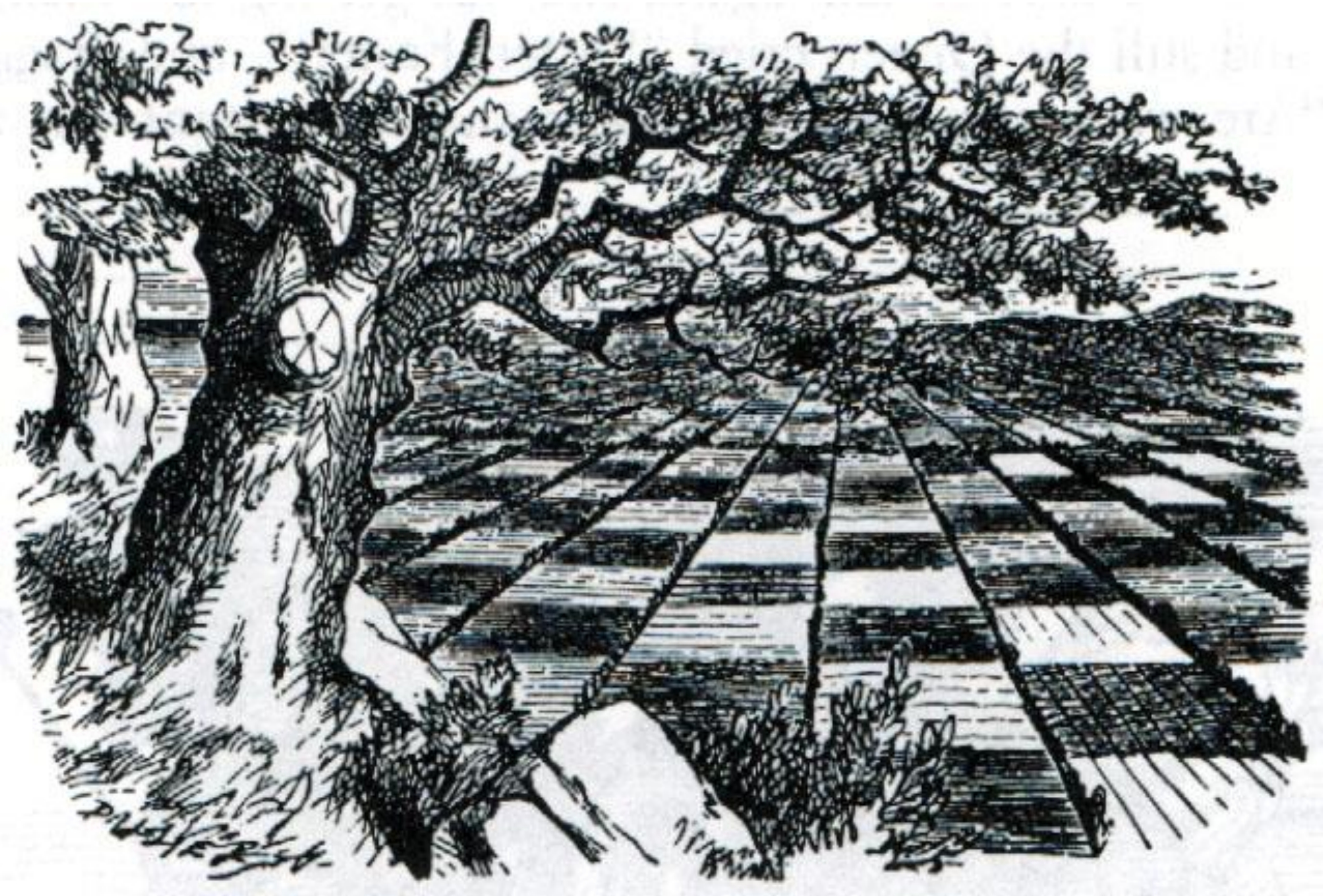
A checkerboard-free, symmetry-preserving, conservative method, for magnetohydrodynamic flows

J.A. Hopman, F.X. Trias, J. Rigola



UNIVERSITAT POLITÈCNICA DE CATALUNYA
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EUROTHERM
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Bled, Slovenia

Motivation

MagnetoHydroDynamic (MHD) flows for Nuclear Fusion

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-High Hartmann number,

$$Ha = LB_0\sqrt{\frac{\sigma_0}{\rho_0 \nu}}$$

Motivation

MagnetoHydroDynamic (MHD) flows for Nuclear Fusion

- -High Hartmann number, $Ha = LB_0\sqrt{\frac{\sigma_0}{\rho_0\nu}}$
- -Low magnetic Reynolds number, $Re_m = \sigma\mu L u_0 \ll 1$

Challenges

Complex geometries

Balancing Lorentz force & pressure drop

Challenges

Complex geometries

→ Collocated grids

Balancing Lorentz force & pressure drop

Challenges

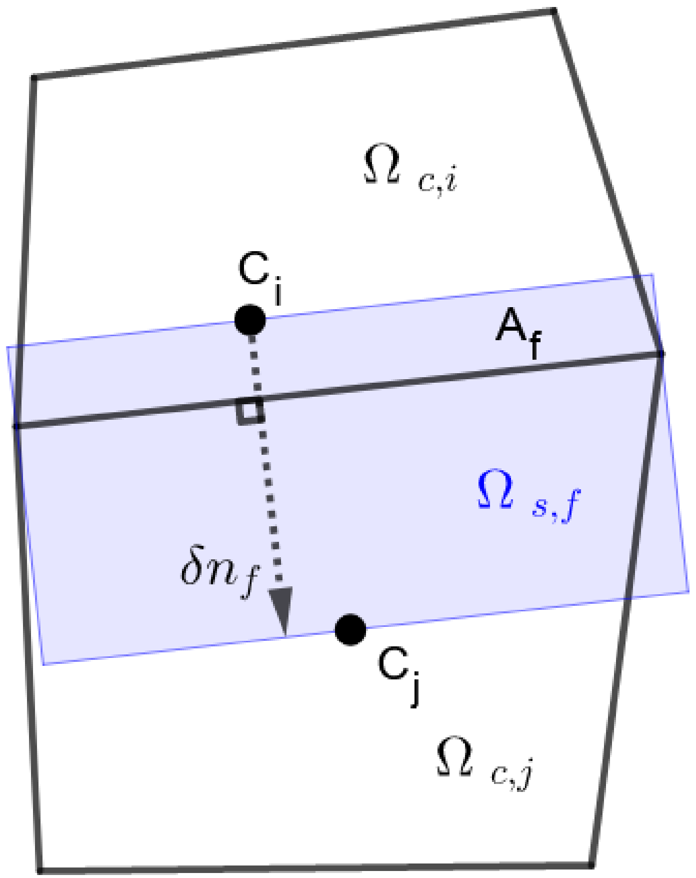
Complex geometries

→ Collocated grids

Balancing Lorentz force & pressure drop

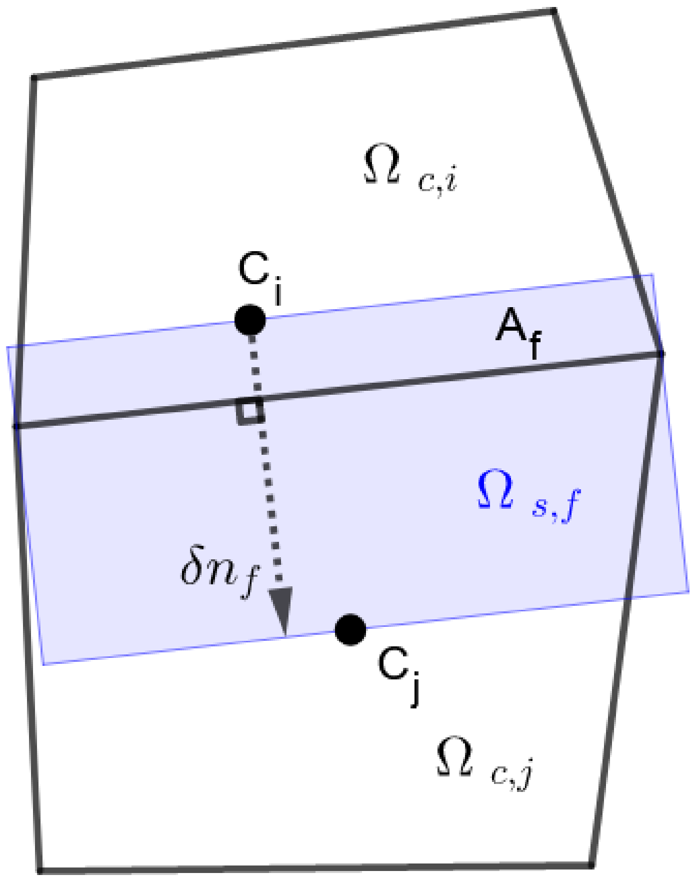
→ Symmetry-preserving method

Symmetry-Preserving method



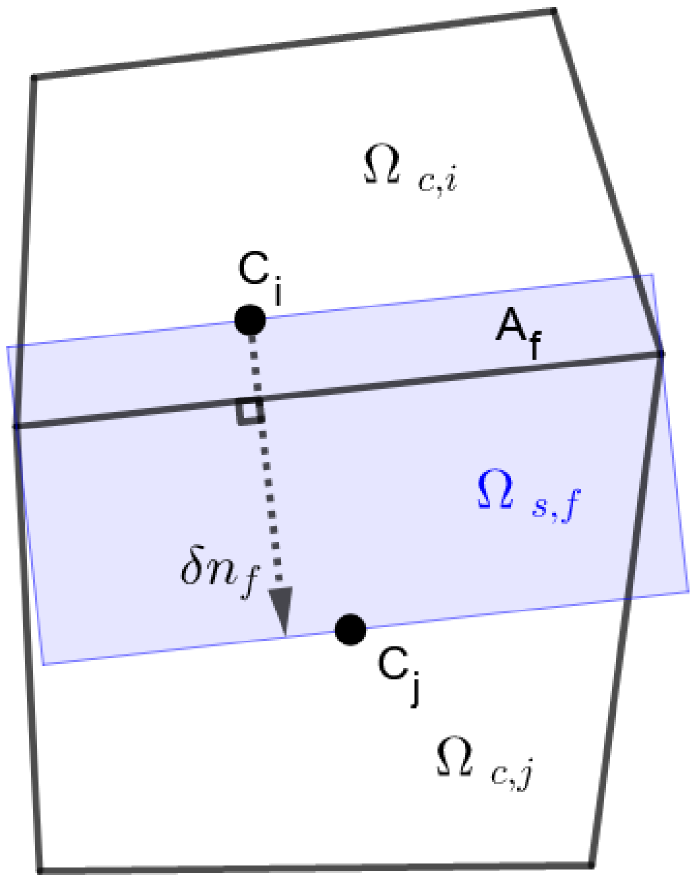
Symmetry-Preserving method

1. Projected gradient distances



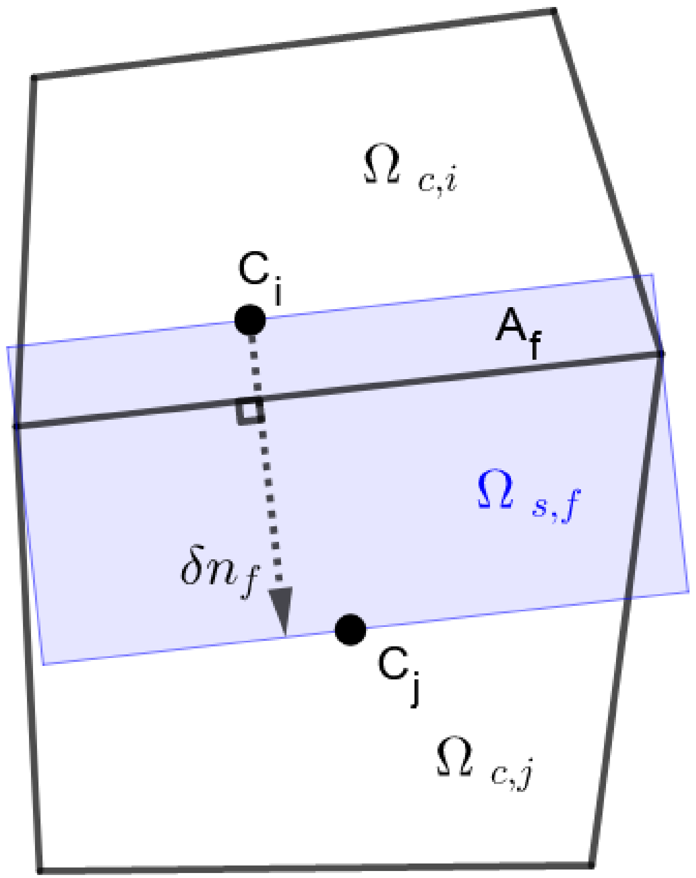
Symmetry-Preserving method

1. Projected gradient distances
2. Consistent Div, Grad, Lap



Symmetry-Preserving method

1. Projected gradient distances
2. Consistent Div, Grad, Lap
3. Midpoint interpolation in C($\mathbf{u}_s$)



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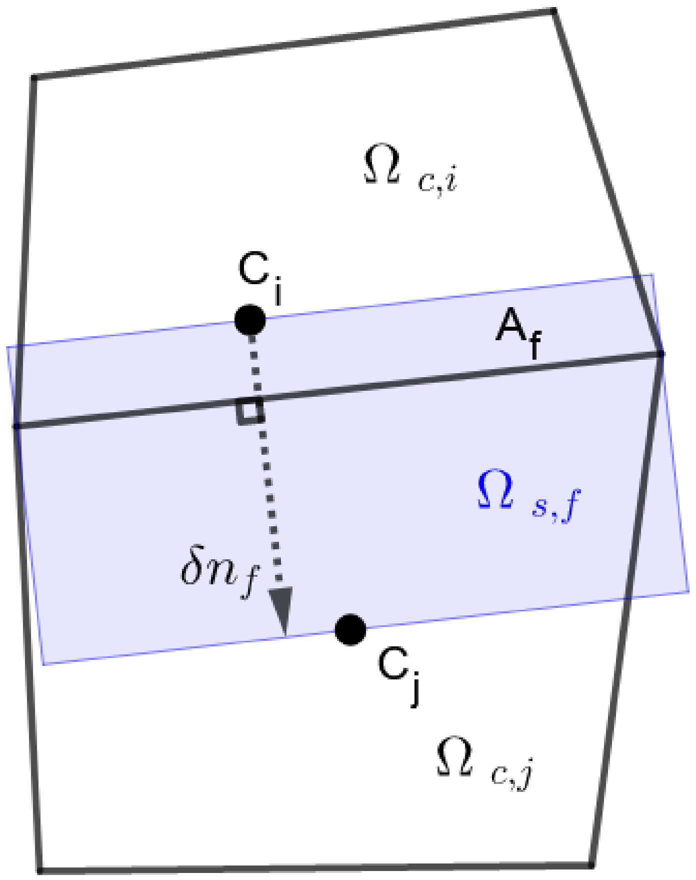
4. Volumetric interpolation



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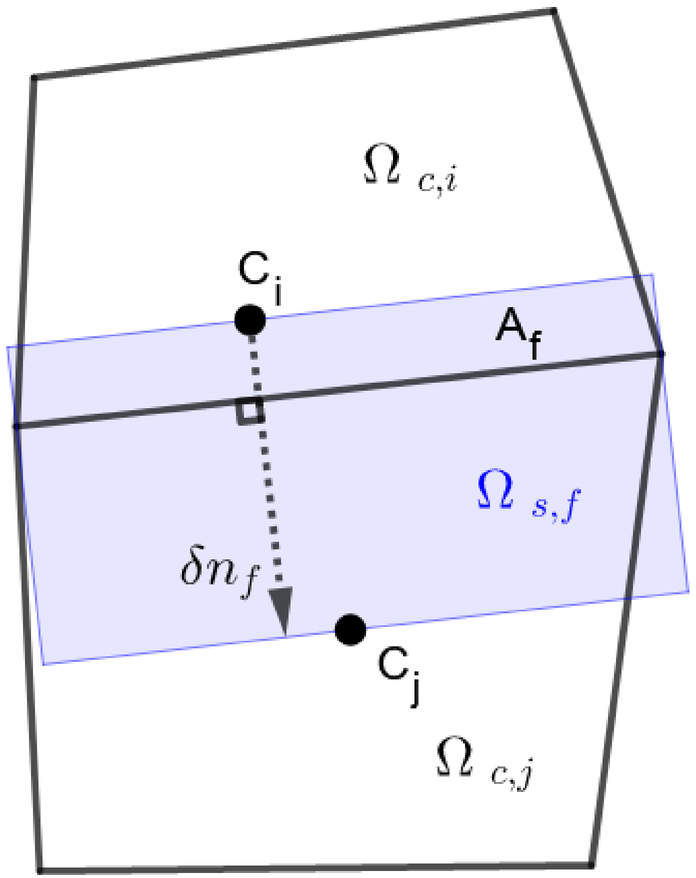
   - Flux term of Poisson equation



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3. Midpoint interpolation in $C(\mathbf{u}_s)$
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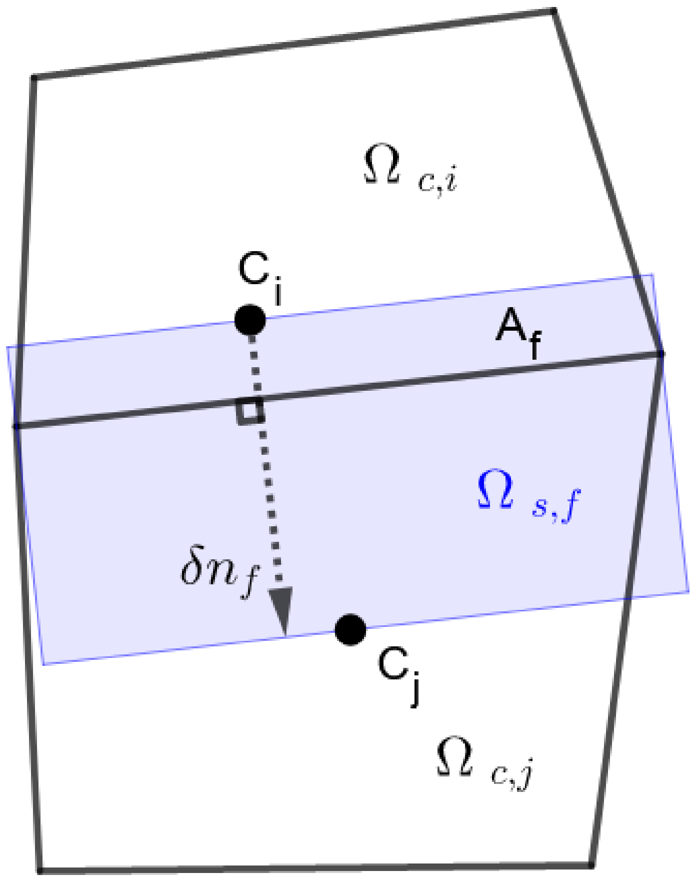
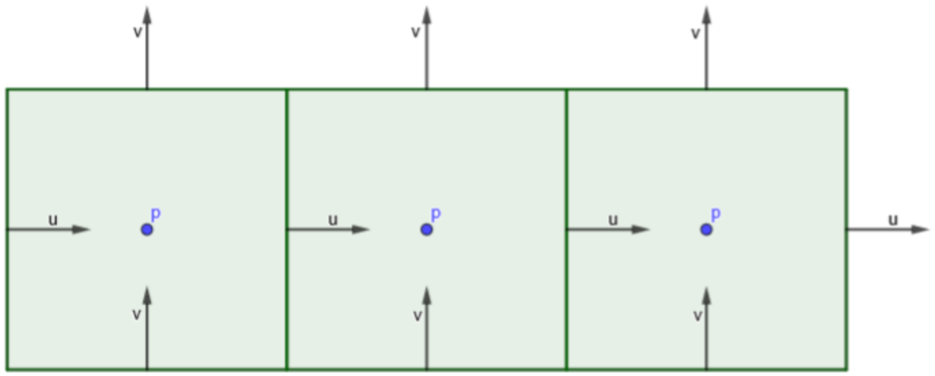
   - Flux term of Poisson equation
   - Correction term after Poisson



Checkerboarding



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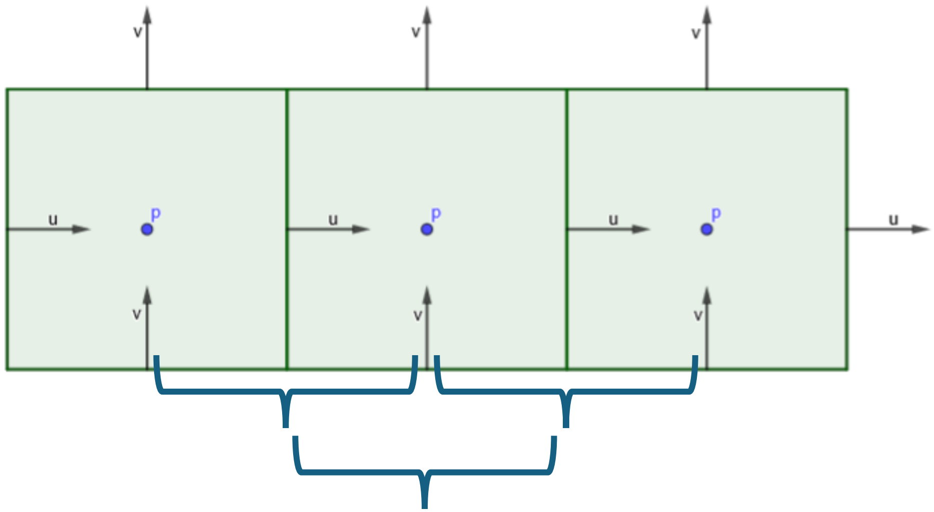
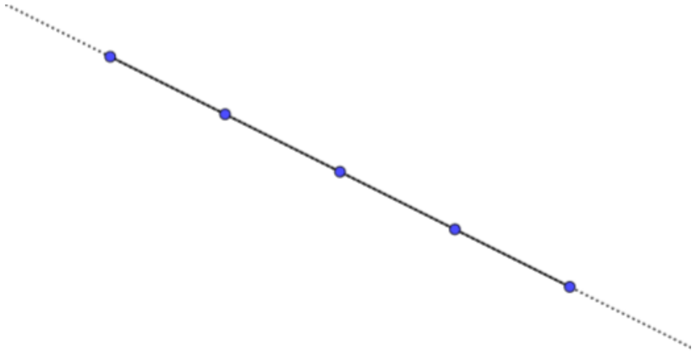


$L = MG$ compact stencil



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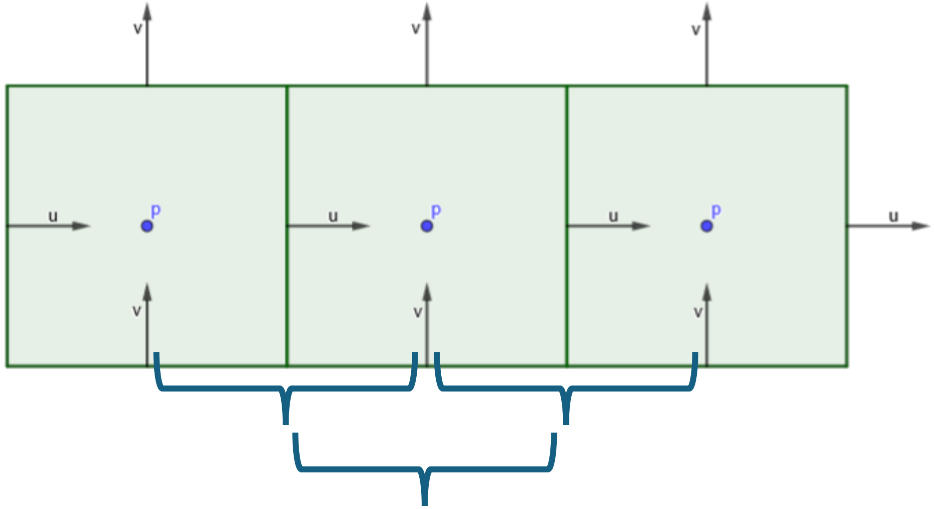

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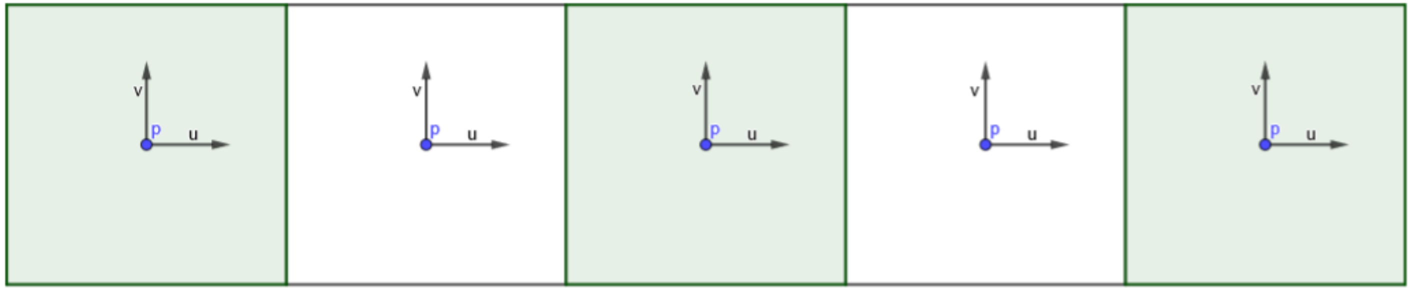

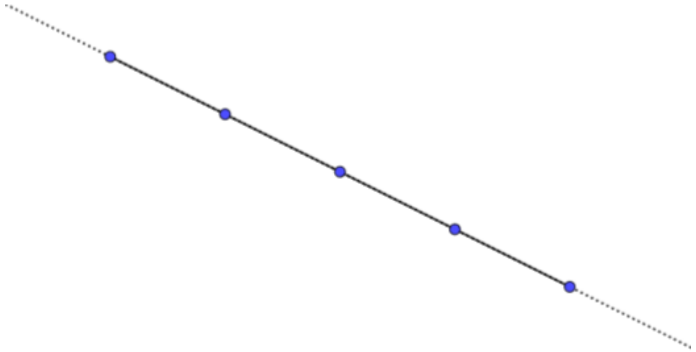

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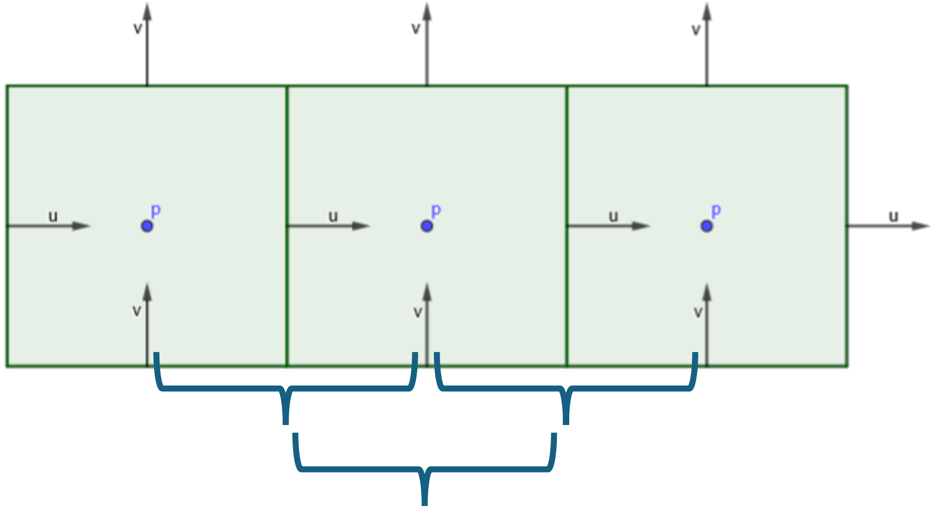

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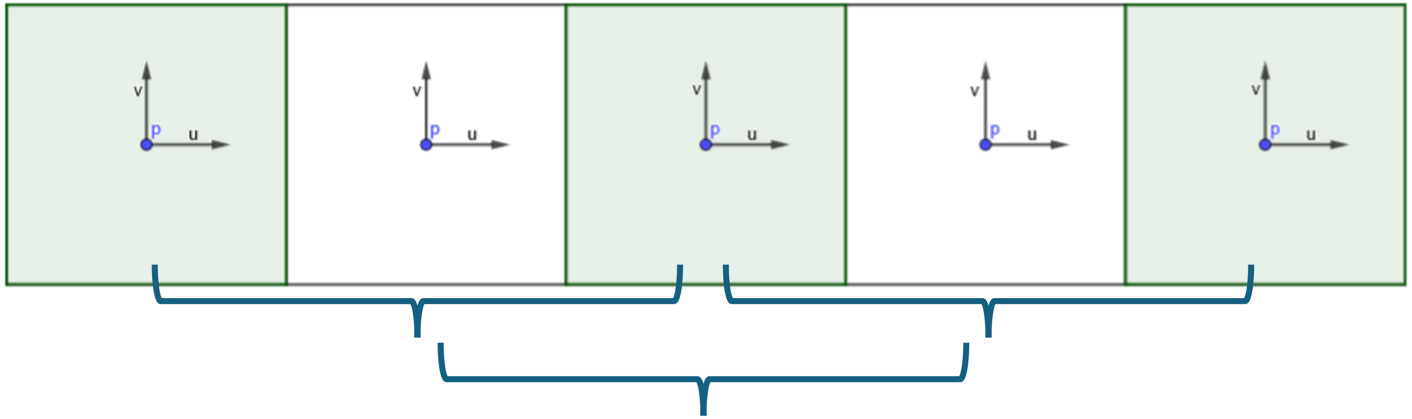
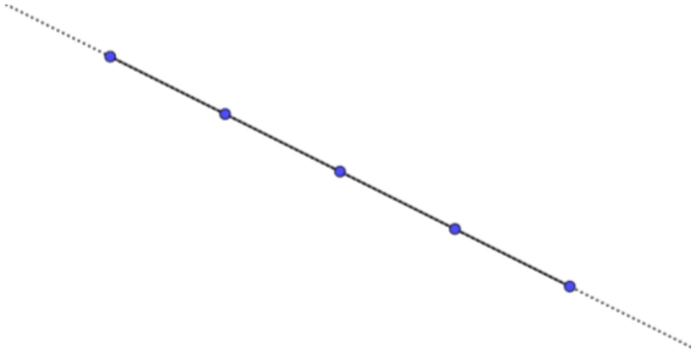
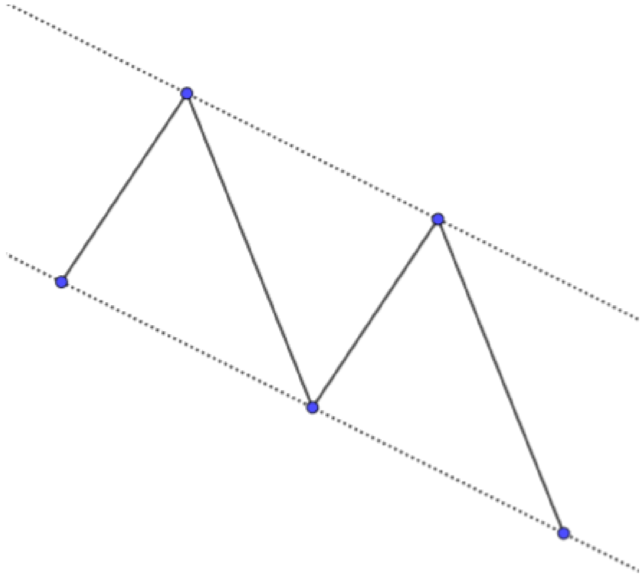


$L_c = M_c G_c$ wide stencil → decoupling





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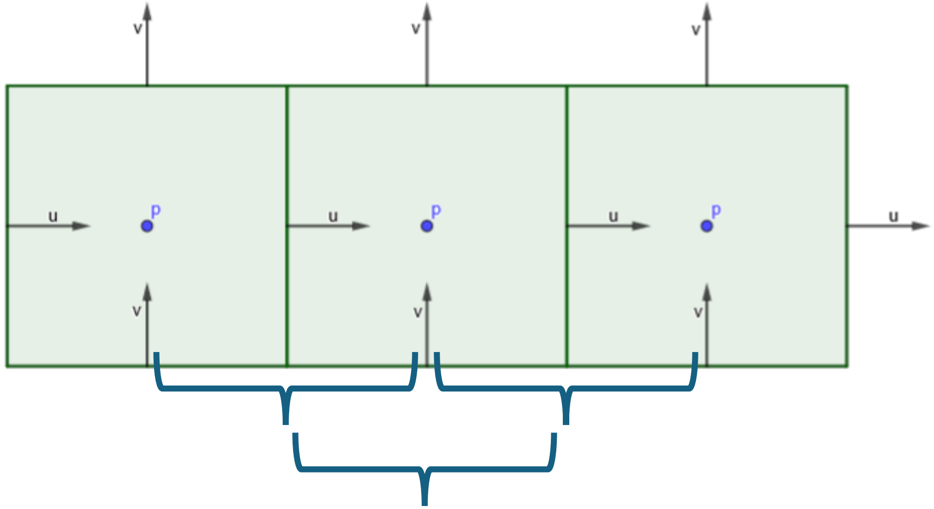

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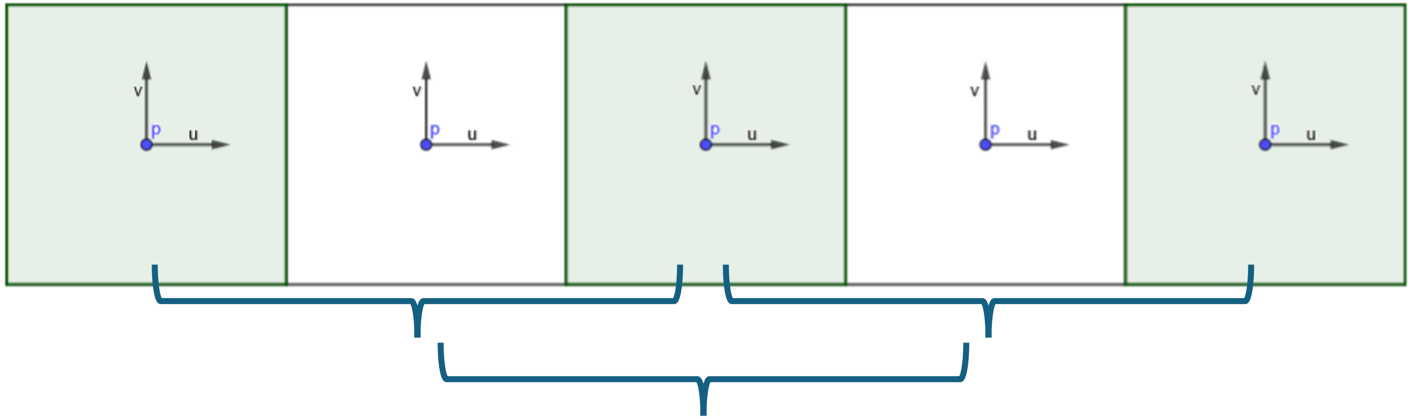

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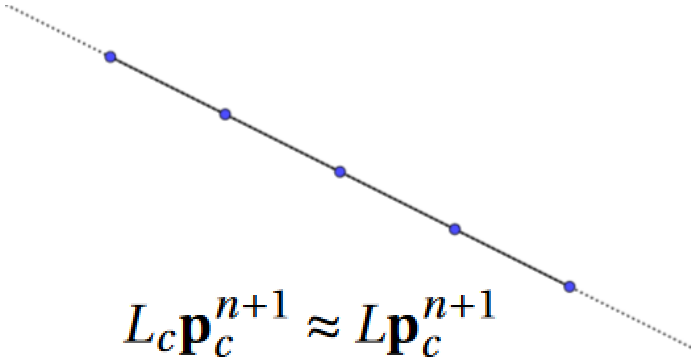
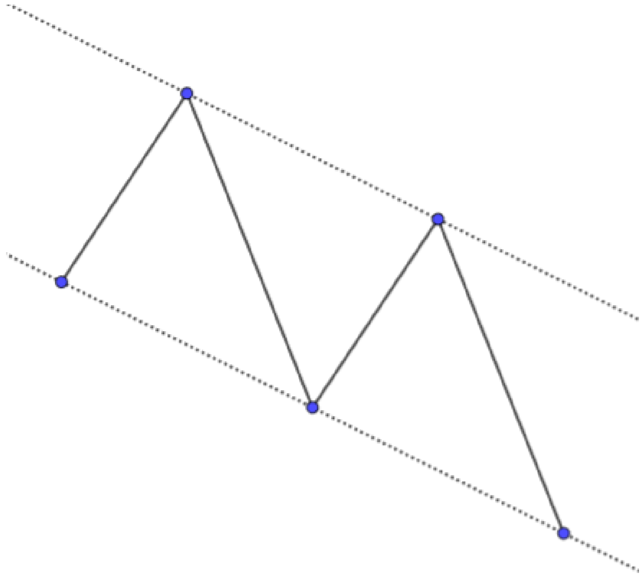


$L_c \mathbf{p}_c^{n+1} \approx L \mathbf{p}_c^{n+1}$

→ pressure error



Pressure prediction

$$\mathbf{u}_c^{p*} = \mathbf{u}_c^p - G_c \tilde{\mathbf{p}}_c^p$$

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$$\tilde{\mathbf{p}}_c^{n+1} = \tilde{\mathbf{p}}_c^p + \tilde{\mathbf{p}}_c'$$

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Lowering pressure error ~ $(L - L_c)\,\tilde{\mathbf{p}}_c'$

Pressure prediction

$$\mathbf{u}_c^{p*} = \mathbf{u}_c^{p} - G_c \tilde{\mathbf{p}}_c^{p}$$

$$L \tilde{\mathbf{p}}'_c = M_c \mathbf{u}_c^{p*} = M_c \mathbf{u}_c^{p} - L_c \tilde{\mathbf{p}}_c^{p}$$

$$\tilde{\mathbf{p}}_c^{n+1} = \tilde{\mathbf{p}}_c^{p} + \tilde{\mathbf{p}}'_c$$

Lowering pressure error ~ $(L - L_c)\,\tilde{\mathbf{p}}'_c$

Larger part on L_c → More prone to checkerboarding

Induction-less approximation

Formulation of second Poisson equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = \nu\nabla^2\mathbf{u} - \nabla\left(p/\rho\right) + \left(\mathbf{J}\times\mathbf{B}\right)/\rho, \qquad \nabla\cdot\mathbf{u} = 0,$$

$$\mathbf{J} = \sigma\left(-\nabla\phi + \mathbf{u}\times\mathbf{B}\right), \qquad \nabla\cdot\mathbf{J} = 0.$$

Quantifying checkerboarding

Starting from the pressure budget term:

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$$-\mathbf{u}_c^T \Omega G_c \mathbf{p}_c$$

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Starting from the pressure budget term:

$$-\mathbf{u}_c^T \Omega G_c \mathbf{p}_c = \mathbf{p}_c^T M_c \mathbf{u}_c = \Delta t \mathbf{p}_c^T (L - L_c) \mathbf{p}_c$$

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Rearranging to:

$$C_{cb} = 1 - \frac{\mathbf{p}_c^T L_c \mathbf{p}_c}{\mathbf{p}_c^T L \mathbf{p}_c}$$

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Rearranging to:

$$C_{cb} = 1 - \frac{\mathbf{p}_c^T L_c \mathbf{p}_c}{\mathbf{p}_c^T L \mathbf{p}_c} = 1 - \frac{\mathbf{p}_c^T G_c^T \Omega G_c \mathbf{p}_c}{\mathbf{p}_c^T G^T \Omega_s G \mathbf{p}_c}$$

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Balance checkerboarding and accuracy

General predictor coefficient:

$$\theta_a = \frac{a_c^T L_c a_c}{a_c^T L a_c}$$

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Low Cb

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Low Cb → Use more predictor

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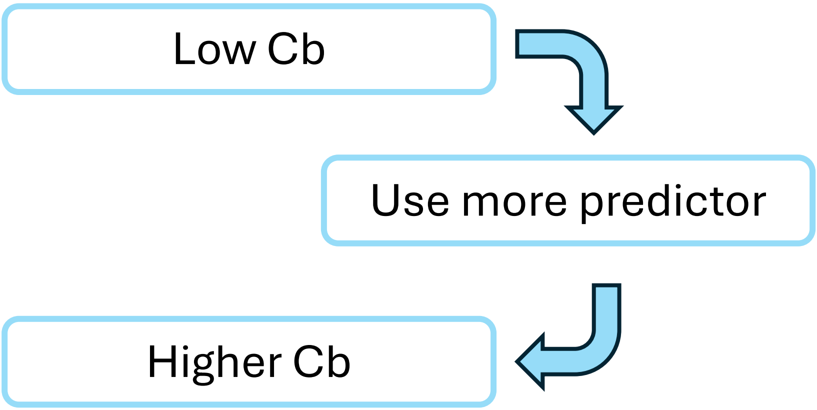

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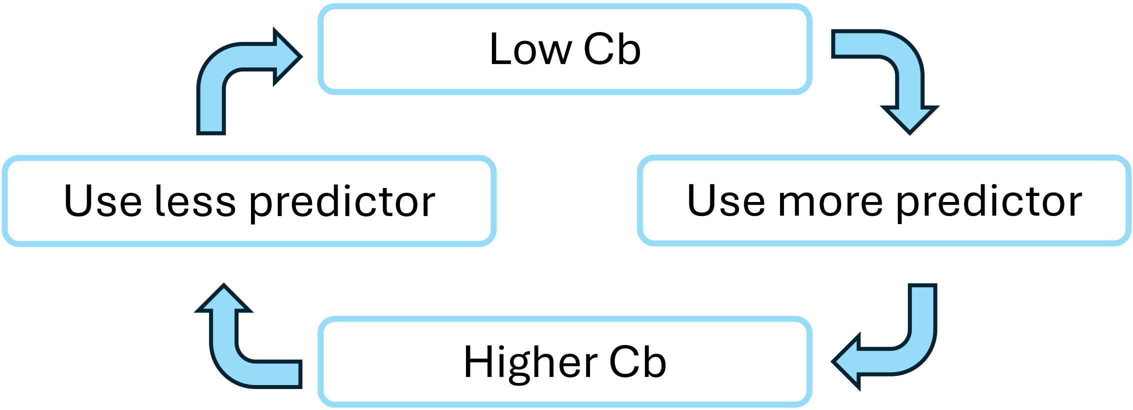
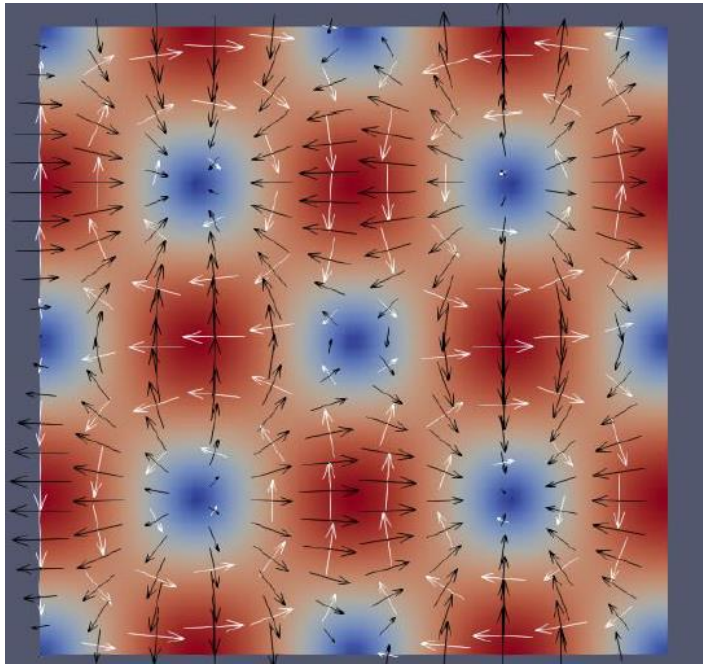


Magnetic 2D Taylor-Green Vortex



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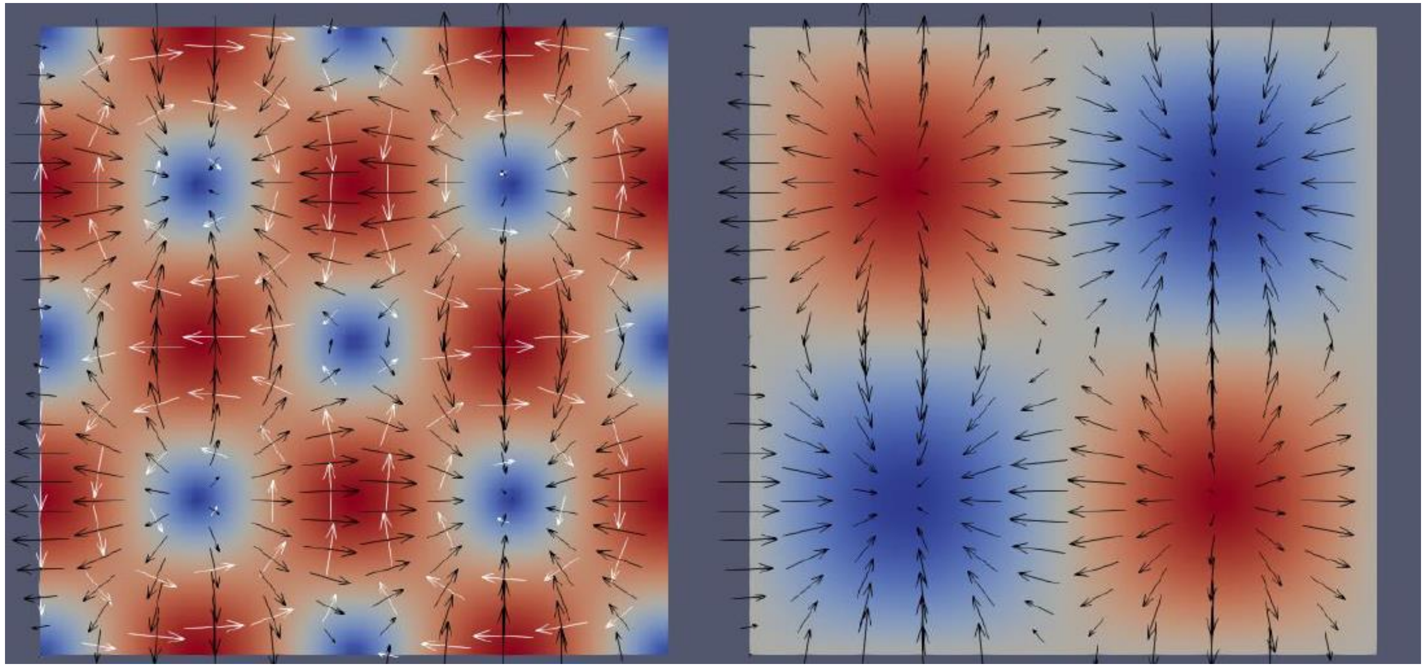

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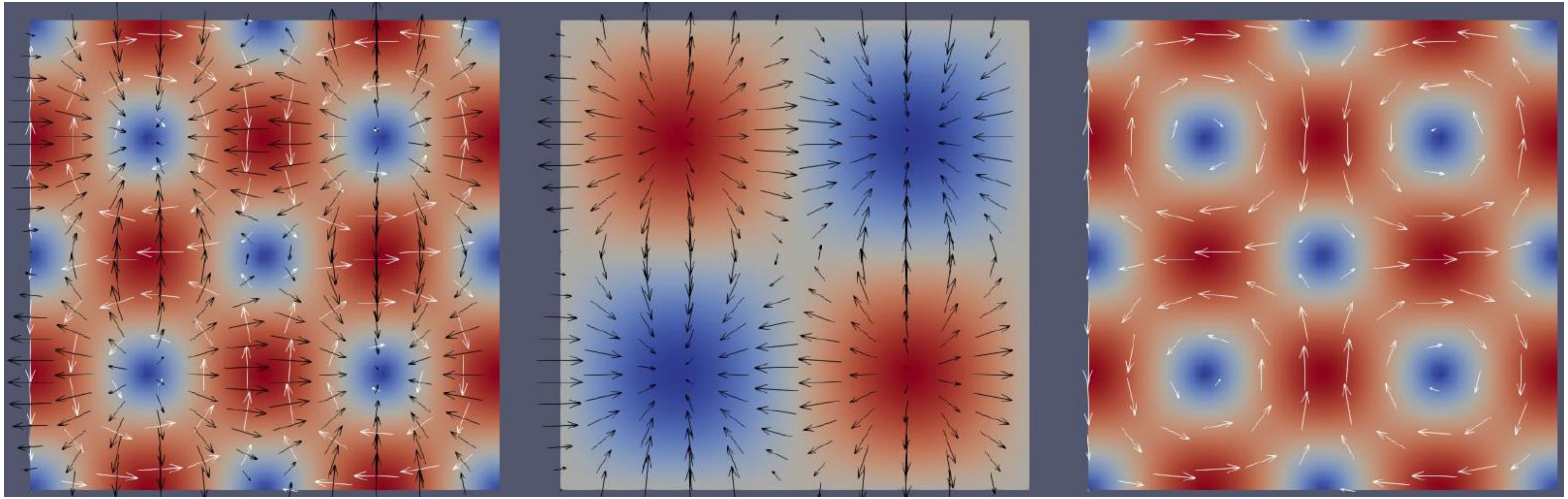

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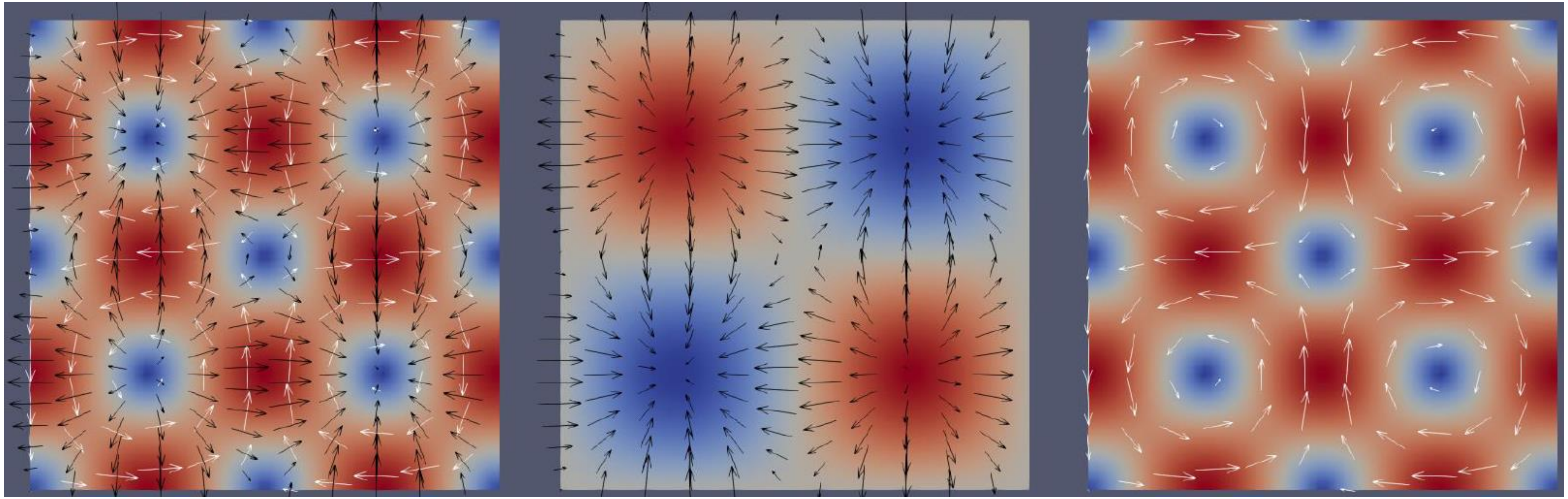

Solvers: SP-θ_0 SP-θ_1 SP-θ_{dy} Ni

Ni M J, Munipalli R, Morley N B, Huang P and Abdou M A 2007 *J. Comput. Phys.* **227** 174–204
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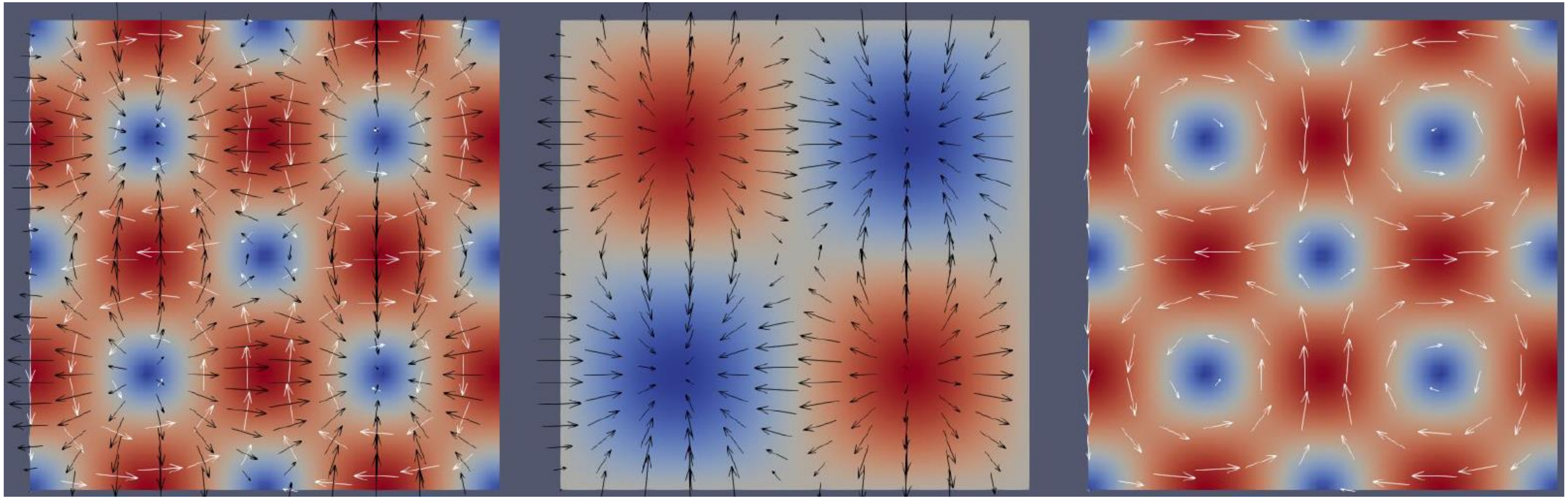

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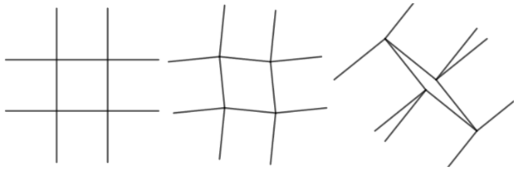
Meshes:



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Error analysis

Kinetic energy budgets

$\partial_t E_k$	Evolution
$= -\mathbf{u} \cdot ((\mathbf{u} \cdot \nabla)\, \mathbf{u})$	Transport
$- \mathbf{u} \cdot (\nabla p) / \rho$	Pressure diffusion
$+ \nu \nabla^2 E_k$	Viscous diffusion
$- \nu (\nabla \mathbf{u}) : (\nabla \mathbf{u})$	Dissipation
$+ \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) / \rho,$	Lorentz force term

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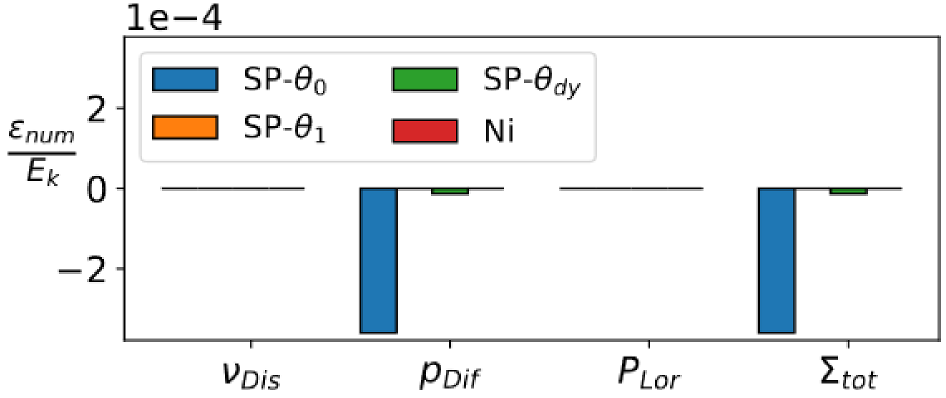
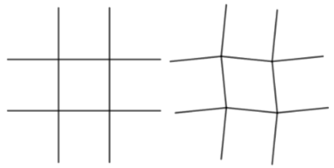
$$\epsilon_{num} = \frac{B_{num} - B_{ana}}{E_k}$$

Accuracy



(a) Inviscid, $Ha = 0$, uniform mesh



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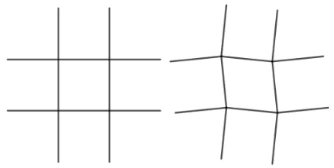

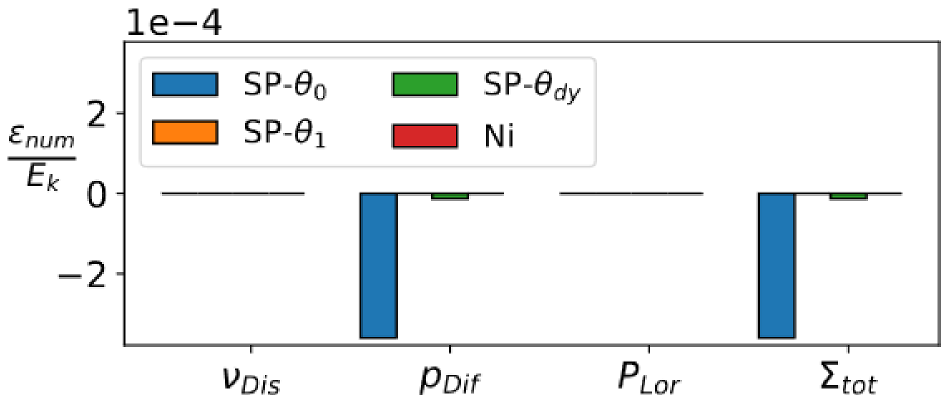

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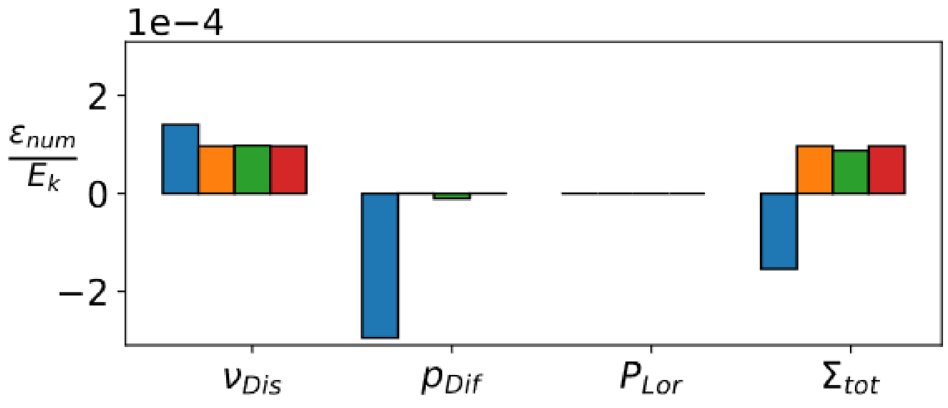
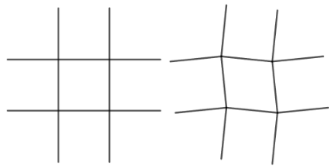
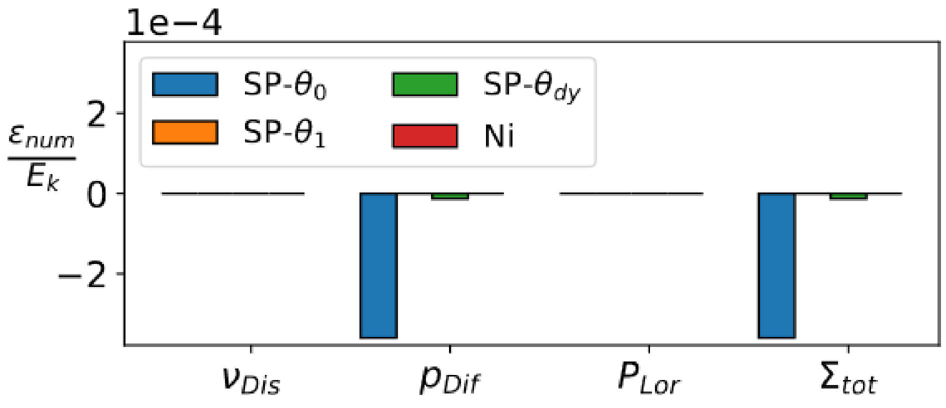


(b) $Re = 100$, $Ha = 0$, uniform mesh.

Accuracy





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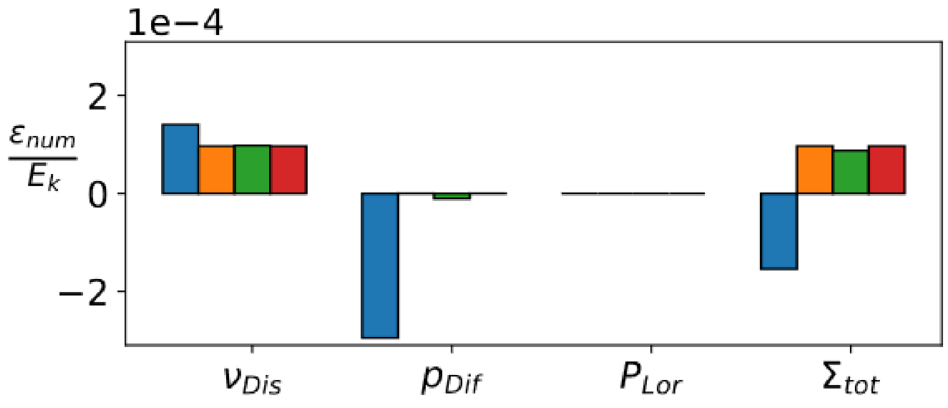

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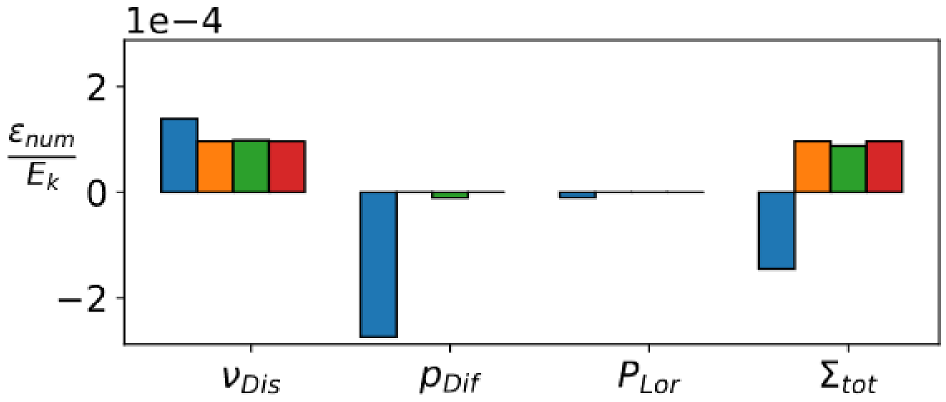
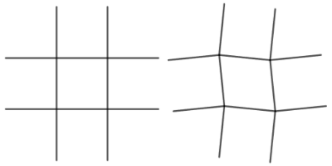
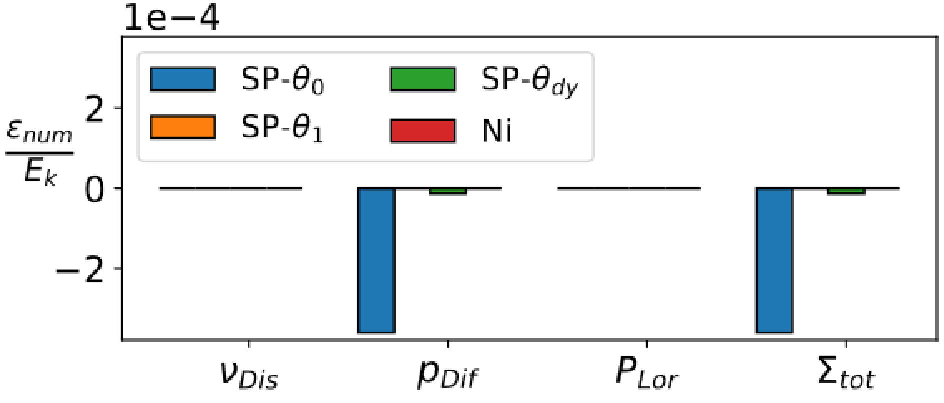


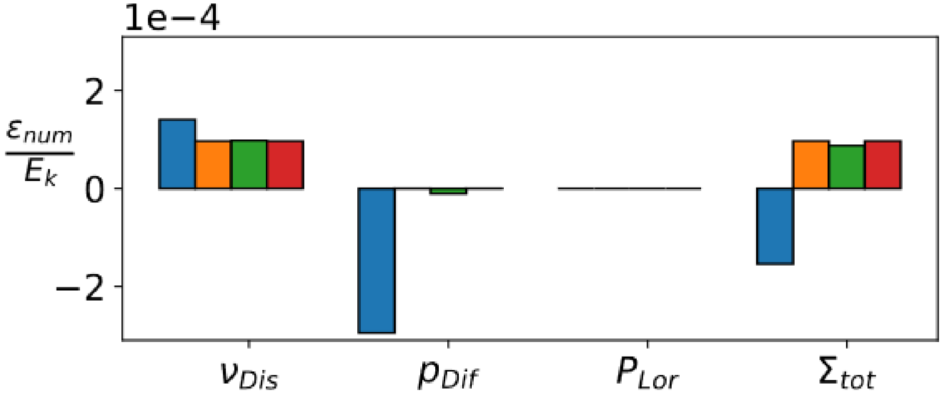
(c) $Re = 100$, $Ha = 100$, uniform mesh

Accuracy

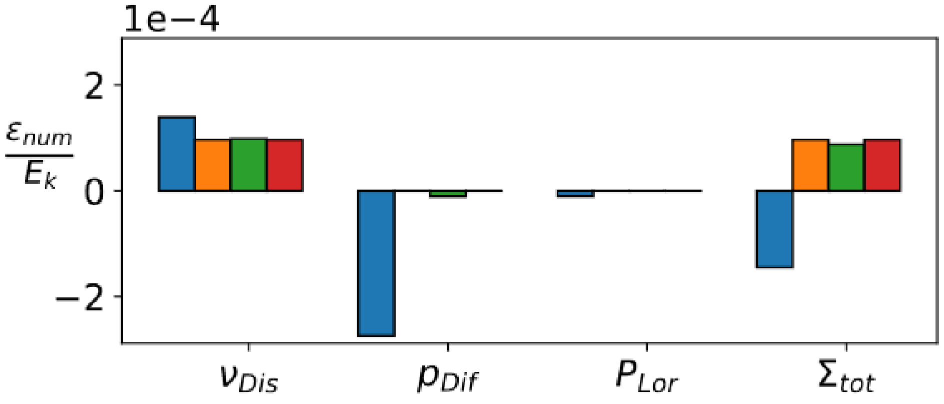




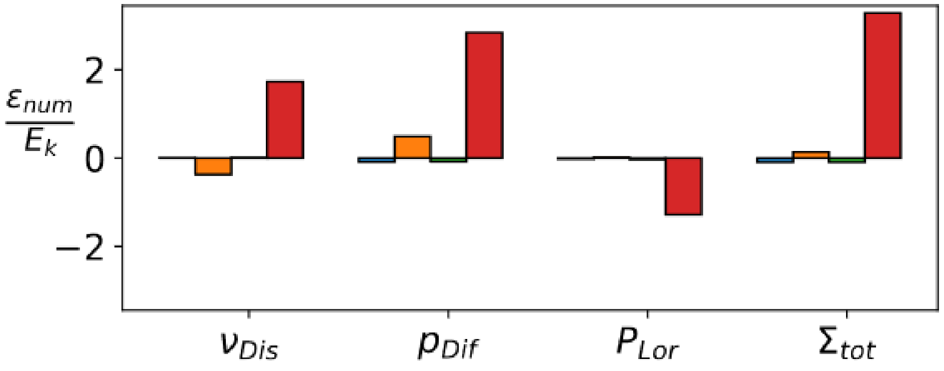
(a) Inviscid, $Ha = 0$, uniform mesh



(b) $Re = 100$, $Ha = 0$, uniform mesh.

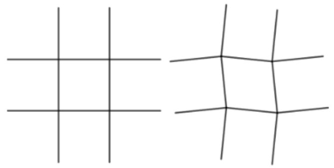


(c) $Re = 100$, $Ha = 100$, uniform mesh



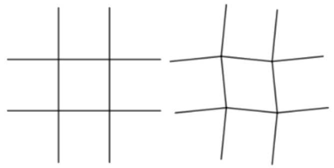
(d) $Re = 100$, $Ha = 100$, perturbed mesh.

Checkerboarding



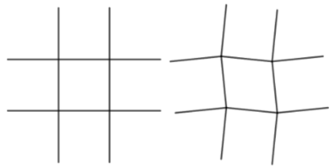
			SP-θ_0		SP-θ_1		SP-θ_{dy}		Ni	
Re	Ha	Mesh	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}
Inviscid	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	100	uniform	0.04	0.01	0.04	0.01	0.04	0.01	0.04	0.01
100	100	perturbed	0.83	0.01	1.00	0.89	0.85	0.01	1.00	0.05

Checkerboarding



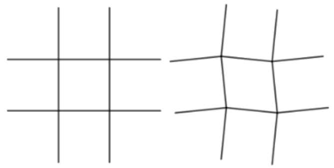
			SP-θ_0		SP-θ_1		SP-θ_{dy}		Ni	
Re	Ha	Mesh	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}
Inviscid	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	100	uniform	0.04	0.01	0.04	0.01	0.04	0.01	0.04	0.01
100	100	perturbed	0.83	0.01	1.00	0.89	0.85	0.01	1.00	0.05

Checkerboarding



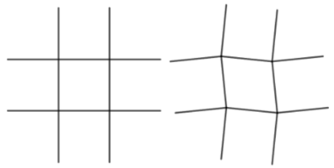
			SP-θ_0		SP-θ_1		SP-θ_{dy}		Ni	
Re	Ha	Mesh	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}
Inviscid	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	100	uniform	0.04	0.01	0.04	0.01	0.04	0.01	0.04	0.01
100	100	perturbed	0.83	0.01	1.00	0.89	0.85	0.01	1.00	0.05

Checkerboarding



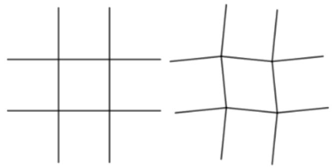
			SP-θ_0		SP-θ_1		SP-θ_{dy}		Ni	
Re	Ha	Mesh	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}
Inviscid	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	100	uniform	0.04	0.01	0.04	0.01	0.04	0.01	0.04	0.01
100	100	perturbed	0.83	0.01	1.00	0.89	0.85	0.01	1.00	0.05

Checkerboarding



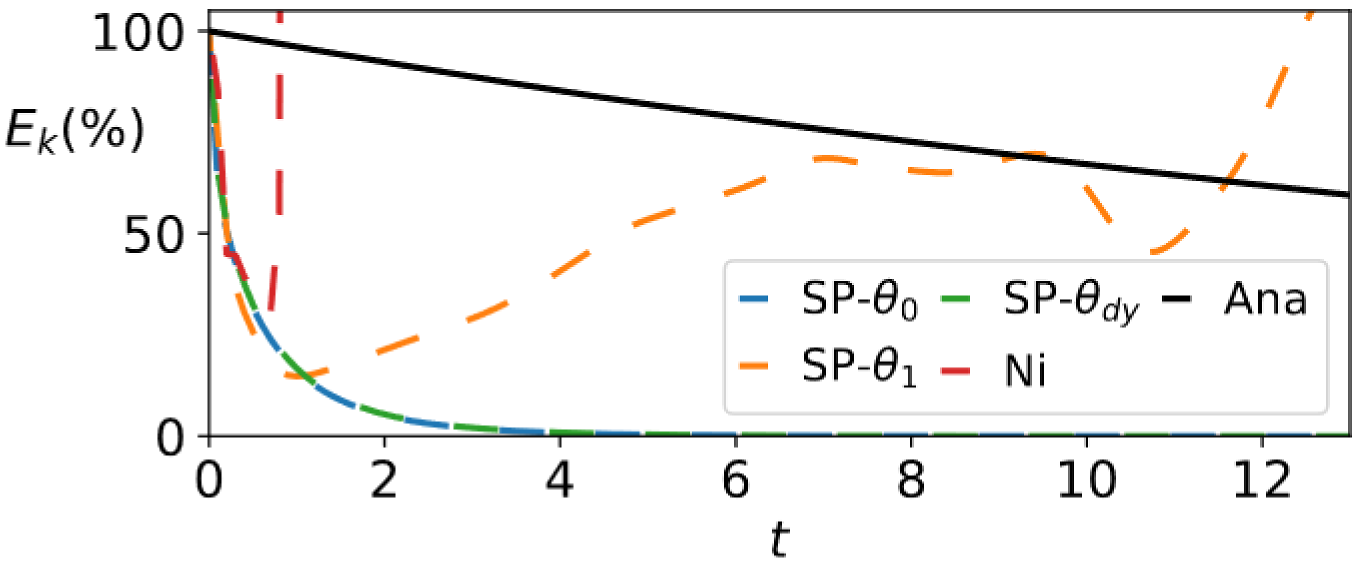
			SP-θ_0		SP-θ_1		SP-θ_{dy}		Ni	
Re	Ha	Mesh	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}
Inviscid	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	100	uniform	0.04	0.01	0.04	0.01	0.04	0.01	0.04	0.01
100	100	perturbed	0.83	0.01	1.00	0.89	0.85	0.01	1.00	0.05

Checkerboarding



			SP-θ_0		SP-θ_1		SP-θ_{dy}		Ni	
Re	Ha	Mesh	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}	p_{cb}	ϕ_{cb}
Inviscid	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	0	uniform	0.04	-	0.04	-	0.04	-	0.04	-
100	100	uniform	0.04	0.01	0.04	0.01	0.04	0.01	0.04	0.01
100	100	perturbed	0.83	0.01	1.00	0.89	0.85	0.01	1.00	0.05

Stability



Conclusions

-Effective method to quantify checkerboarding

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-Balances checkerboarding and numerical dissipation

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-Turns “on” on perturbed meshes → eliminate checkerboard

Conclusions

-Effective method to quantify checkerboarding

-Balances checkerboarding and numerical dissipation

- -Turns "off" on uniform meshes → eliminate dissipation
- -Turns "on" on perturbed meshes → eliminate checkerboard

-Symmetry-preserving method is robust