Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

UNIVERSITÀ DEGLI STUDI DI PADOVA

On the evolution of Poisson solvers

F.Xavier Trias[1], Àdel Alsalti-Baldellou[1,2]

[1]Heat and Mass Transfer Technological Center, Technical University of Catalonia
[2]Department of Civil, Environmental and Architectural Engineering, University of Padova

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On the evolution of Poisson solvers for extreme-scale CFD simulations

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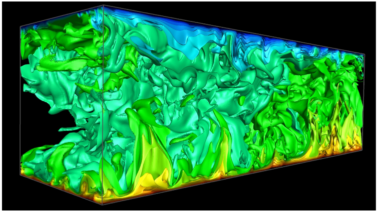
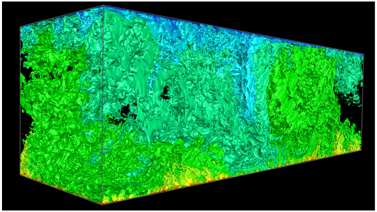
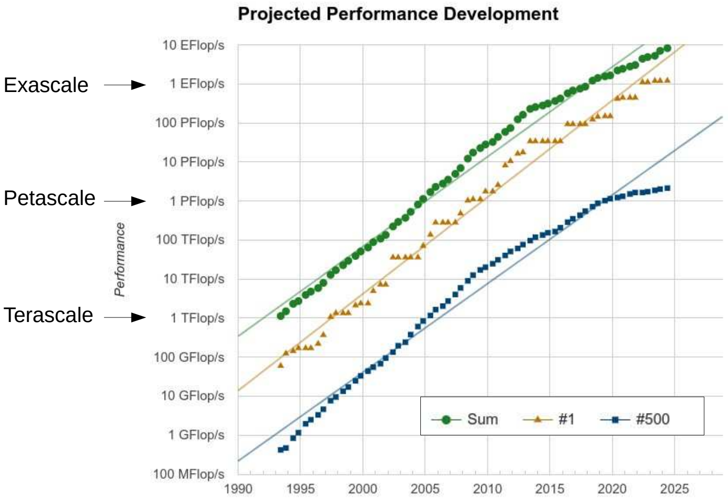




Contents

Tera, Peta, Exa,..., Zetta, Yotta

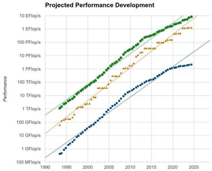


Source: `www.top500.org`

Tera, Peta, Exa,..., Zetta, Yotta

~10 years

	PetaFLOPS		#1 in LINPACK	#1 in HPCG	Cutting-edge CFD simulation	'Routine' CFD simulation
Zetta	10^6					
Exa	10^3	14 years	**2022** (Frontier)			
Peta	1		**2008** (Roadrunner)	**2018** (Summit)		
Tera	10^{-3}	11 years	**1997** (ASCI Red)	No data		



Tera, Peta, Exa,..., Zetta, Yotta

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	PetaFLOPS		#1 in LINPACK	#1 in HPCG	Cutting-edge CFD simulation	'Routine' CFD simulation
Zetta	10^6		2037	2047		
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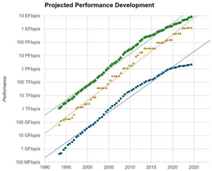

Tera, Peta, Exa,..., Zetta, Yotta

~10 years ~5 years

	PetaFLOPS		#1 in LINPACK	#1 in HPCG	Cutting-edge CFD simulation	'Routine' CFD simulation
Zetta	10^6		**2037**	**2047**	**2052**	
Exa	10^3	14 years	**2022** (Frontier)	**2032**	**2037**	
Peta	1		**2008** (Roadrunner)	**2018** (Summit)	**2023**	
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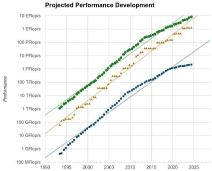
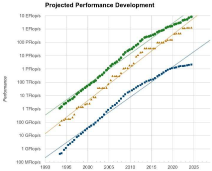


Tera, Peta, Exa,..., Zetta, Yotta

~10 years | ~5 years | ~10 years

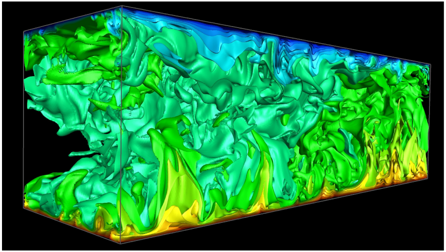
	PetaFLOPS		#1 in LINPACK	#1 in HPCG	Cutting-edge CFD simulation	'Routine' CFD simulation
Zetta	10^6		2037	2047	2052	2062
Exa	10^3	14 years	**2022** (Frontier)	2032	2037	2047
Peta	1		**2008** (Roadrunner)	**2018** (Summit)	2023	2033
Tera	10^{-3}	11 years	**1997** (ASCI Red)	No data		

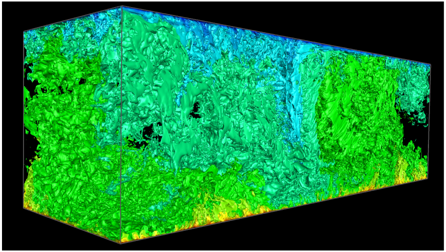


Motivation

Research question :

- Will the **complexity** of numerically solving **Poisson's equation** increase or decrease for **very large scale DNS/LES** simulations of incompressible turbulent flows?

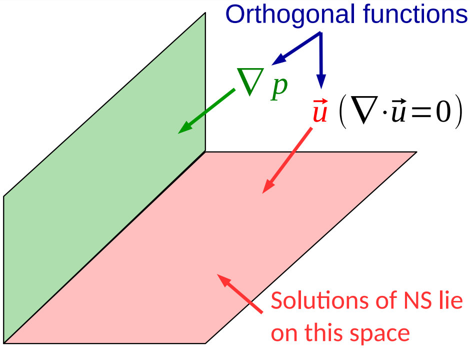




DNS[1] of air-filled Rayleigh–Bénard convection at $Ra = 10^8$ and 10^{10}

[1] B.Sanderse, F.X.Trias. *Energy-consistent discretization of viscous dissipation with application to natural convection flow*. **Computers & Fluids**, 286:106473, 2025

Poisson's equation: a quick reminder

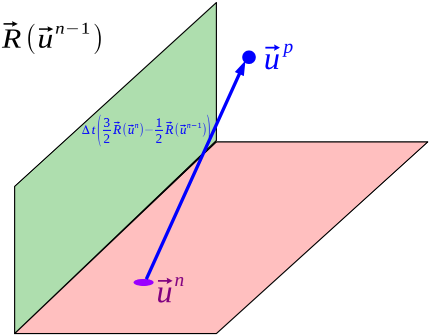


Semi-discrete (just in time) NS equations

$$\begin{cases} \dfrac{\vec{u}^{n+1}-\vec{u}^{n}}{\Delta t}=\dfrac{3}{2}\vec{R}(\vec{u}^{n})-\dfrac{1}{2}\vec{R}(\vec{u}^{n-1})-\nabla p^{n+1} \\ \nabla\cdot\vec{u}^{n+1}=0 \end{cases}$$

Poisson's equation: a quick reminder

Step 1: $\dfrac{\vec{u}^{p}-\vec{u}^{n}}{\Delta t}=\dfrac{3}{2}\vec{R}(\vec{u}^{n})-\dfrac{1}{2}\vec{R}(\vec{u}^{n-1})$



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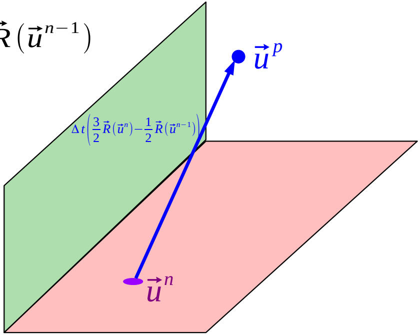
$$\begin{cases}\dfrac{\vec{u}^{n+1}-\vec{u}^{n}}{\Delta t}=\dfrac{3}{2}\vec{R}(\vec{u}^{n})-\dfrac{1}{2}\vec{R}(\vec{u}^{n-1})-\nabla p^{n+1}\\ \nabla\cdot\vec{u}^{n+1}=0\end{cases}$$

Poisson's equation: a quick reminder

Step 1: $\dfrac{\vec{u}^{\,p}-\vec{u}^{\,n}}{\Delta t}=\dfrac{3}{2}\vec{R}(\vec{u}^{\,n})-\dfrac{1}{2}\vec{R}(\vec{u}^{\,n-1})$

Step 2: $\boxed{\nabla^2 p^{n+1}=\dfrac{1}{\Delta t}\nabla\cdot\vec{u}^{\,p}}$



Semi-discrete (just in time) NS equations

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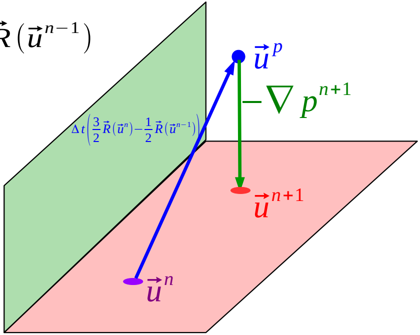
Poisson's equation: a quick reminder

Step 1: $\frac{\vec{u}^{p}-\vec{u}^{n}}{\Delta t}=\frac{3}{2}\vec{R}(\vec{u}^{n})-\frac{1}{2}\vec{R}(\vec{u}^{n-1})$

Step 2: $\boxed{\nabla^2 p^{n+1}=\frac{1}{\Delta t}\nabla\cdot\vec{u}^{p}}$

Step 3: $\vec{u}^{n+1}=\vec{u}^{p}-\Delta t\,\nabla p^{n+1}$



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Poisson's equation: getting more tough or not?

Research question:

- Will the **complexity** of numerically solving **Poisson's equation** **increase** or **decrease** for **very large scale DNS/LES** simulations of incompressible turbulent flows?

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^{\,p}}$$

Two competing effects: who (if any) will eventually win?

$$\mathrm{Re}\uparrow \;\to\; \Delta x \downarrow \;\longrightarrow\; N_x \uparrow \;\longrightarrow\; \text{Larger system} \;\downarrow$$

$$\mathrm{Re}\uparrow \;\to\; \Delta t \downarrow \;\longrightarrow\; \text{Better initial guess} \;\uparrow$$

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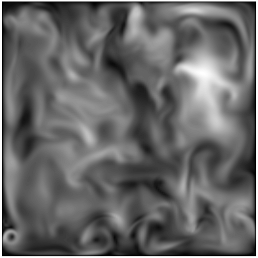
$Ra = 10^8$



[2] F.Dabbagh, F.X.Trias, A.Gorobets, A.Oliva. *Flow topology dynamics in a 3D phase space for turbulent Rayleigh-Bénard convection*, **Phys.Rev.Fluids**, 5:024603, 2020.

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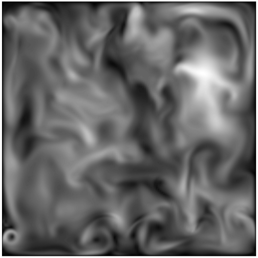
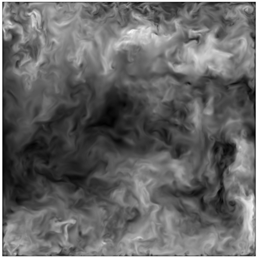
$Ra = 10^8$



$Ra = 10^{10}$



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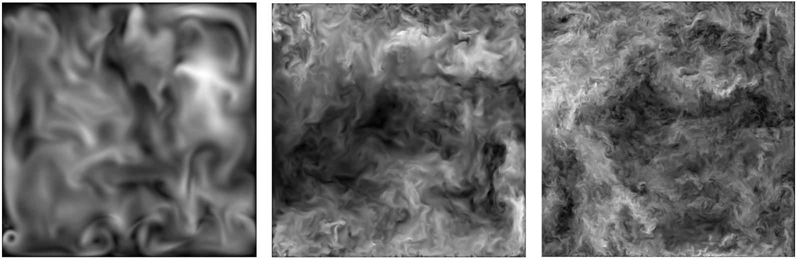
$Ra = 10^8$ $Ra = 10^{10}$ $Ra = 10^{11}$



[2]F.Dabbagh, F.X.Trias, A.Gorobets, A.Oliva. *Flow topology dynamics in a 3D phase space for turbulent Rayleigh-Bénard convection*, **Phys.Rev.Fluids**, 5:024603, 2020.

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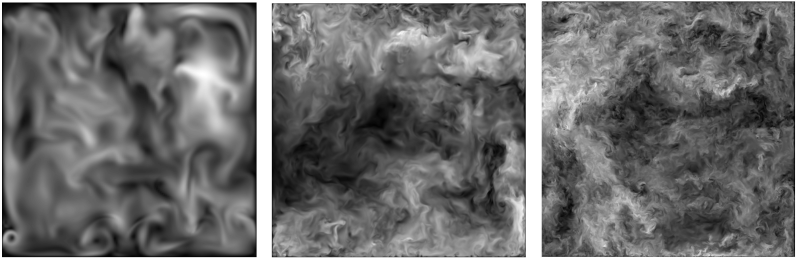
$Ra = 10^8$ | $Ra = 10^{10}$ | $Ra = 10^{11}$



$208 \times 208 \times 400$ **17.5M** | $768 \times 768 \times 1024$ **607M** | $1662 \times 1662 \times 2048$ **5600M**

[2] F.Dabbagh, F.X.Trias, A.Gorobets, A.Oliva. *Flow topology dynamics in a 3D phase space for turbulent Rayleigh-Bénard convection*, **Phys.Rev.Fluids**, 5:024603, 2020.

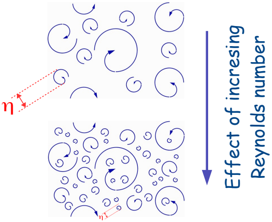
Smaller and smaller, but how much?

Two competing effects: who (if any) will eventually win?

$$\mathrm{Re}\uparrow \nearrow \Delta x \downarrow \longrightarrow N_x \uparrow \longrightarrow \text{Larger system} \downarrow$$

$$\mathrm{Re}\uparrow \searrow \Delta t \downarrow \longrightarrow \text{Better initial guess} \uparrow$$

l: biggest eddies (driving scale)
η: smallest eddies (Kolmogorov length scale)



Question: ?

how $\frac{\eta}{l}$ decreases with Re?

From classical K41 theory:

$$\frac{1}{N_x^{\mathrm{K41}}} = \frac{\Delta x}{L_x} \sim \frac{\eta}{l} \propto \mathrm{Re}^{-3/4}$$

$$\frac{u}{U} \propto \mathrm{Re}^{-1/4}$$

$$\frac{1}{N_t^{\mathrm{K41}}} = \frac{\Delta t}{t_{\mathrm{sim}}} \sim \frac{t_\eta}{t_l} \propto \frac{\eta}{l}\frac{U}{u} \propto \mathrm{Re}^{-3/4}\,\mathrm{Re}^{1/4} = \mathrm{Re}^{-1/2}$$

Smaller and smaller, but how much?

Two competing effects: who (if any) will eventually win?

$$\mathrm{Re}\uparrow \;\rightarrow\; \Delta x \downarrow \;\rightarrow\; N_x \uparrow \;\rightarrow\; \text{Larger system} \downarrow$$

$$\mathrm{Re}\uparrow \;\rightarrow\; \Delta t \downarrow \;\rightarrow\; \text{Better initial guess} \uparrow$$

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From CFL condition:

$$\Delta t^{\mathrm{conv}} \sim \frac{\Delta x}{U} \qquad \Delta t^{\mathrm{diff}} \sim \frac{\Delta x^2}{\nu}$$

$$\frac{1}{N_t^{\mathrm{conv}}} \sim \frac{\Delta t^{\mathrm{conv}}}{t_l} \sim \frac{U}{l}\frac{l\,\mathrm{Re}^{-3/4}}{U} = \mathrm{Re}^{-3/4}$$

$$\frac{1}{N_t^{\mathrm{diff}}} \sim \frac{\Delta t^{\mathrm{diff}}}{t_l} \sim \frac{U}{l}\frac{l^2\left(\mathrm{Re}^{-3/4}\right)^2}{\nu} = \mathrm{Re}^{-1/2}$$

Smaller and smaller, but how much?

Two competing effects: who (if any) will eventually win?

$$\mathrm{Re}\uparrow \;\to\; \Delta x\downarrow \;\to\; N_x\uparrow \;\to\; \text{Larger system } \color{red}{\downarrow}$$

$$\mathrm{Re}\uparrow \;\to\; \Delta t\downarrow \;\to\; \text{Better initial guess } \color{blue}{\uparrow}$$

In summary:

$$\frac{1}{N_x^{\mathrm{K41}}} = \frac{\Delta x}{L_x} \sim \frac{\color{red}{\eta}}{\color{blue}{l}} \propto \mathrm{Re}^{-3/4}$$

$$\frac{\Delta t}{\color{blue}{t_l}} \sim \mathrm{Re}^{\alpha}$$

$\alpha = -1/2$ (K41 or diffusion dominated)

$\alpha = -3/4$ (convection dominated)

Residual of Poisson's equation

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^{p}}$$

Initial guess → p^n

$$r^o = \nabla^2 p^n - \frac{1}{\Delta t} \nabla \cdot u^{p,n+1}$$

Residual of Poisson's equation

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^{p}}$$

Initial guess → p^n

$$r^o = \nabla^2 p^n - \frac{1}{\Delta t} \nabla \cdot u^{p,n+1} = \frac{1}{\Delta t} \nabla \cdot u^{p,n} - \frac{1}{\Delta t} \nabla \cdot u^{p,n+1} \approx \frac{\partial \nabla \cdot u^p}{\partial t} = \nabla \cdot \frac{\partial u^p}{\partial t}$$

Residual of Poisson's equation

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^{\,p}}$$

$$\Big\downarrow \text{ Initial guess} \rightarrow \; p^n$$

$$r^o = \nabla^2 p^n - \frac{1}{\Delta t} \nabla \cdot u^{p,n+1} = \frac{1}{\Delta t} \nabla \cdot u^{p,n} - \frac{1}{\Delta t} \nabla \cdot u^{p,n+1} \approx \frac{\partial \nabla \cdot u^p}{\partial t} = \nabla \cdot \frac{\partial u^p}{\partial t}$$

$$\tilde{r}^o = \nabla^2 \tilde{p}^n - \nabla \cdot u^{p,n+1} \approx \nabla \cdot u^{p,n} - \nabla \cdot u^{p,n+1} \approx \Delta t \frac{\partial \nabla \cdot u^p}{\partial t} = \Delta t \nabla \cdot \frac{\partial u^p}{\partial t}$$

$$\Big\uparrow \text{ Initial guess} \rightarrow \; \tilde{p}^n = \Delta t \, p^n$$

$$\boxed{\nabla^2 \tilde{p}^{n+1} = \nabla \cdot \vec{u}^{\,p}}$$

Residual of Poisson's equation

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^{\,p}}$$

$\downarrow$ Initial guess → p^n

$$r^o \approx \frac{\partial \nabla \cdot u^p}{\partial t}$$

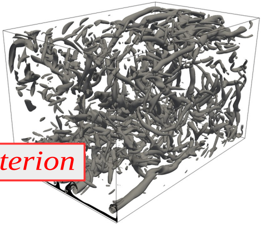
$$\widetilde{r}^o \approx \Delta t \frac{\partial \nabla \cdot u^p}{\partial t}$$

$\uparrow$ Initial guess → $\widetilde{p}^n = \Delta t\, p^n$

$$\boxed{\nabla^2 \widetilde{p}^{n+1} = \nabla \cdot \vec{u}^{\,p}}$$

Residual of Poisson's equation

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^p}$$



$Q_G - criterion$

Initial guess → p^n

$$r^o \approx \frac{\partial \nabla \cdot u^p}{\partial t}$$

What is $\nabla \cdot u^p$?

$$\nabla \cdot u^p = \cancel{\nabla \cdot u^n} - \Delta t \nabla \cdot (u^n \cdot \nabla u^n) \cancel{+ \nu \Delta t \nabla \cdot \nabla^2 u^n} = 2 \Delta t \, Q_G$$

$$\tilde{r}^o \approx \Delta t \frac{\partial \nabla \cdot u^p}{\partial t}$$

$$\boxed{Q_G = -\frac{1}{2} tr(G^2) \quad \text{where} \quad G = \nabla u^n}$$

Initial guess → $\tilde{p}^n = \Delta t \, p^n$

$$\boxed{\nabla^2 \tilde{p}^{n+1} = \nabla \cdot \vec{u}^p}$$

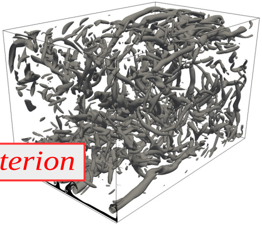
Residual of Poisson's equation

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^{\,p}}$$

Initial guess → p^n

$$r^o \approx \frac{\partial \nabla \cdot u^p}{\partial t} = 2 \Delta t \frac{\partial Q_G}{\partial t}$$

$$\tilde{r}^o \approx \Delta t \frac{\partial \nabla \cdot u^p}{\partial t} = 2 \Delta t^2 \frac{\partial Q_G}{\partial t}$$



$Q_G - criterion$

$$\boxed{R_G = det(G) = \frac{1}{3} tr(G^3)}$$

$$\boxed{Q_G = -\frac{1}{2} tr(G^2) \quad \text{where} \quad G = \nabla u^n}$$

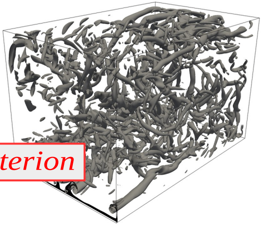
Exact equations for restricted Euler:

$$\frac{d Q_G}{dt} = -3 R_G \longrightarrow \frac{\partial Q_G}{\partial t} = -(u \cdot \nabla) Q_G - 3 R_G$$

Residual of Poisson's equation

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^p}$$

Initial guess → p^n



$$r^o \approx \frac{\partial \nabla \cdot u^p}{\partial t} = 2 \Delta t \frac{\partial Q_G}{\partial t} \approx -2 \Delta t \{(u \cdot \nabla) Q_G + 3 R_G\}$$

$$\tilde{r}^o \approx \Delta t \frac{\partial \nabla \cdot u^p}{\partial t} = 2 \Delta t^2 \frac{\partial Q_G}{\partial t} \approx -2 \Delta t^2 \{(u \cdot \nabla) Q_G + 3 R_G\}$$

Exact equations for restricted Euler:

$$\frac{d Q_G}{d t} = -3 R_G \longrightarrow \frac{\partial Q_G}{\partial t} = -(u \cdot \nabla) Q_G - 3 R_G$$

Residual of Poisson's equation

$$\boxed{\nabla^2 p^{n+1} = \frac{1}{\Delta t}\nabla\cdot\vec{u}^{\,p}}$$

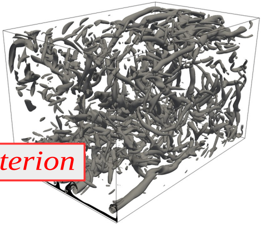
Initial guess → p^n

$$r^o \approx \frac{\partial \nabla\cdot u^p}{\partial t} = 2\Delta t\,\frac{\partial Q_G}{\partial t} \approx -2\Delta t\{(u\cdot\nabla)Q_G + 3R_G\}$$

$$\tilde{r}^o \approx \Delta t\,\frac{\partial \nabla\cdot u^p}{\partial t} = 2\Delta t^2\,\frac{\partial Q_G}{\partial t} \approx -2\Delta t^2\{(u\cdot\nabla)Q_G + 3R_G\}$$

Initial guess → $\tilde{p}^n = \Delta t\, p^n$

$$\boxed{\nabla^2 \tilde{p}^{n+1} = \nabla\cdot\vec{u}^{\,p}}$$



Residual of Poisson's equation in Fourier space

In summary:

$$\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^p$$

$$r^o \approx \frac{\partial \nabla \cdot u^p}{\partial t} = 2 \Delta t^q \frac{\partial Q_G}{\partial t} \approx -2 \Delta t^q \{ (u \cdot \nabla) Q_G + 3 R_G \} \qquad q = \{1, 2\}$$

$$\nabla^2 \tilde{p}^{n+1} = \nabla \cdot \vec{u}^p$$

$$\boxed{\frac{\Delta t}{t_l} \sim \mathrm{Re}^{\alpha} \begin{cases} \alpha = -1/2 & (\text{ K41 or diffusion dominated }) \\ \alpha = -3/4 & (\text{ convection dominated }) \end{cases}}$$

$$\boxed{\frac{1}{N_x^{\mathrm{K41}}} = \frac{\Delta x}{L_x} \sim \frac{\eta}{l} \propto \mathrm{Re}^{-3/4}}$$

Residual of Poisson's equation in Fourier space

In summary:

$$r^o \approx \frac{\partial \nabla \cdot u^p}{\partial t} = 2\Delta t^q \frac{\partial Q_G}{\partial t} \approx -2\Delta t^q \{ (u \cdot \nabla) Q_G + 3R_G \}$$

$\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^p$

$q = \{1, 2\}$

$\nabla^2 \tilde{p}^{n+1} = \nabla \cdot \vec{u}^p$

Hypothesis: (inertial range) $\left(\frac{\partial \hat{Q}_G}{\partial t} \right)_k \propto \mathrm{Re}^{-1} k^{\beta} \longrightarrow \hat{r}^o_k \propto \mathrm{Re}^{-1} \Delta t^q k^{\beta} \sim \mathrm{Re}^{q\alpha - 1} k^{\beta} = \mathrm{Re}^{\tilde{\alpha}} k^{\beta}$

$$\frac{\Delta t}{t_l} \sim \mathrm{Re}^{\alpha} \begin{cases} \alpha = -1/2 & (\text{ K41 or diffusion dominated }) \\ \alpha = -3/4 & (\text{ convection dominated }) \end{cases}$$

$$\frac{1}{N_x^{\mathrm{K41}}} = \frac{\Delta x}{L_x} \sim \frac{\eta}{l} \propto \mathrm{Re}^{-3/4}$$

Residual of Poisson's equation in Fourier space

In summary:

$$\nabla^2 p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{u}^p$$

$$r^o \approx \frac{\partial \nabla \cdot u^p}{\partial t} = 2 \Delta t^q \frac{\partial Q_G}{\partial t} \approx -2 \Delta t^q \{ (u \cdot \nabla) Q_G + 3 R_G \} \qquad q = \{1, 2\}$$

$$\nabla^2 \tilde{p}^{n+1} = \nabla \cdot \vec{u}^p$$

Hypothesis: (inertial range)

$$\left(\frac{\partial \hat{Q}_G}{\partial t} \right)_k \propto \mathrm{Re}^{-1} k^{\beta} \longrightarrow \boxed{\hat{r}_k^o \propto \mathrm{Re}^{-1} \Delta t^q k^{\beta} \sim \mathrm{Re}^{q\alpha - 1} k^{\beta} = \mathrm{Re}^{\tilde{\alpha}} k^{\beta}}$$

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$$\|r\|^2 = \int_{\Omega} r^2 dV = \int_1^{k_{max}} \hat{r}_k^2 dk$$

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Solver convergence

$$\|r^n\|^2 = \int_1^{k_{max}} (\hat{\omega}_k^n \hat{r}_k^0)^2 \, dk \approx \int_1^{C_k \mathrm{Re}^{3/4}} \hat{\omega}_k^{2n} \mathrm{Re}^{2\tilde{\alpha}} k^{2\beta} \, dk$$

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$$\rho \equiv k/k_{max}$$

$$\tilde{\beta} \equiv \frac{3}{4}(\beta + 1/2)$$

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$$\approx C_k^{2\beta+1} \mathrm{Re}^{2(\tilde{\alpha}+\tilde{\beta})} \frac{1}{2} \mathrm{B}(2n+1, \beta+1/2) \qquad \tilde{\beta} \equiv \frac{3}{4}(\beta+1/2)$$

$k_{max} \gg 1$

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$$\underset{n \gg 1}{\approx} C_k^{2\beta+1} \mathrm{Re}^{2(\tilde{\alpha}+\tilde{\beta})} \frac{1}{2} \Gamma(\beta+1/2)(2n+1)^{-(\beta+1/2)}$$

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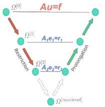

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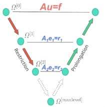

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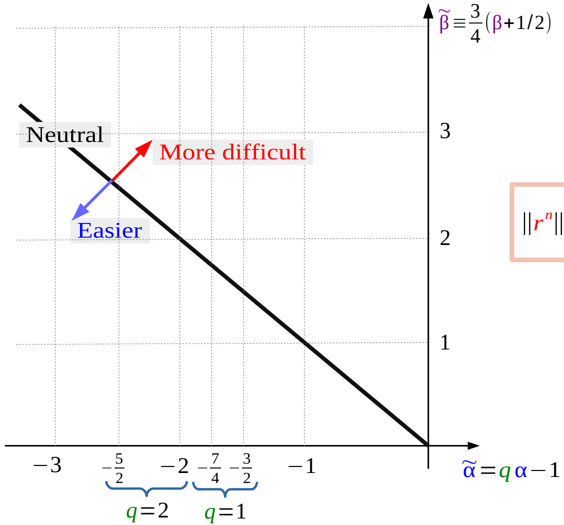
Solver convergence

$\{\tilde{\alpha}, \tilde{\beta}\}$ phase space



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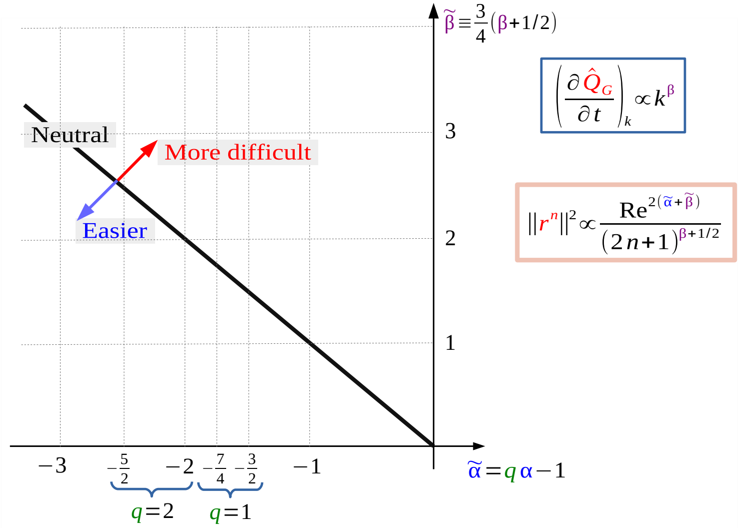

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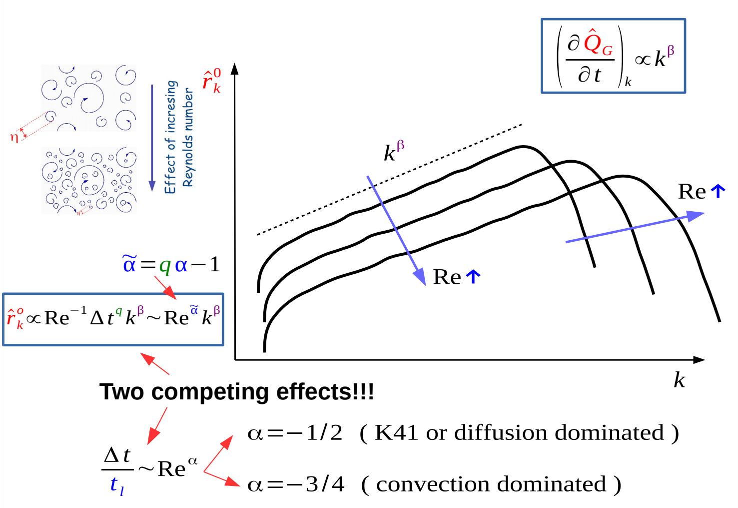

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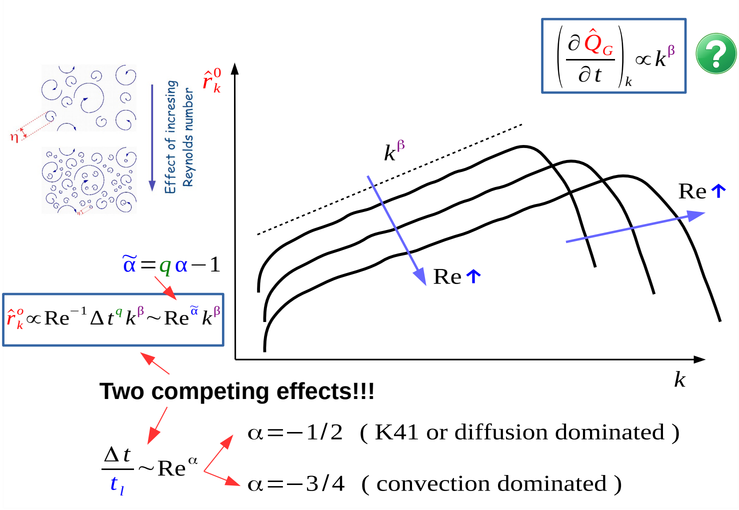

Homogeneous isotropic turbulence

Kolmogorov theory predictions

$$\boxed{E_k = C_k \varepsilon^{2/3} k^{-5/3}}$$

$$\boxed{\left(\frac{\partial \hat{Q}_G}{\partial t}\right)_k \propto k^{\beta}}$$

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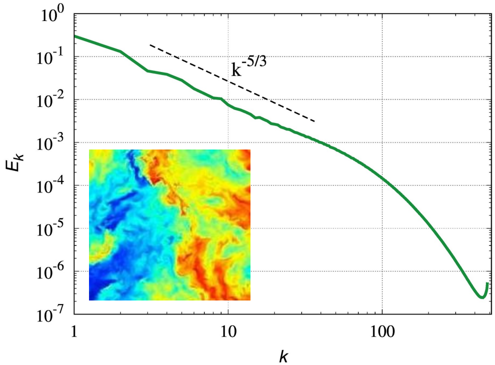

$\mathrm{Re}_\lambda \approx 433$ (1024^3) from https://turbulence.pha.jhu.edu/

Homogeneous isotropic turbulence

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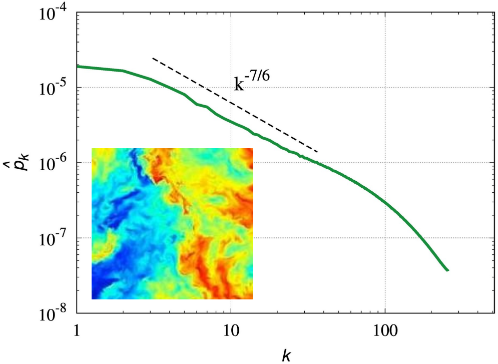

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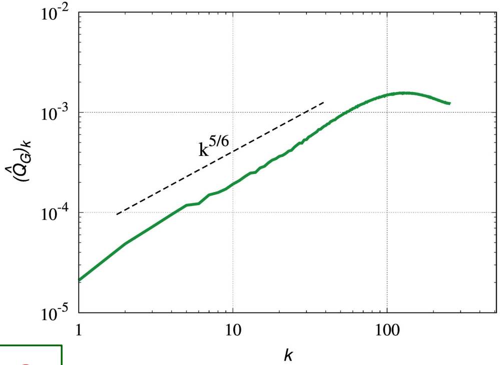

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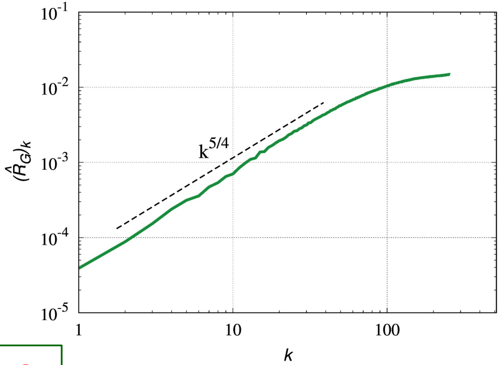

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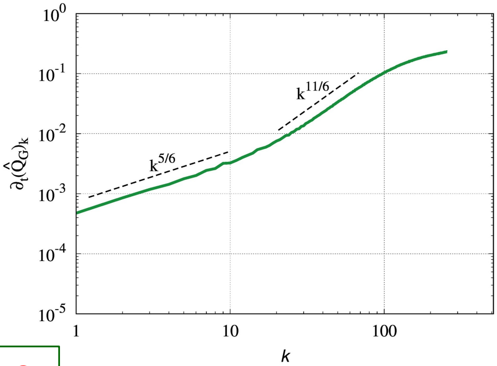

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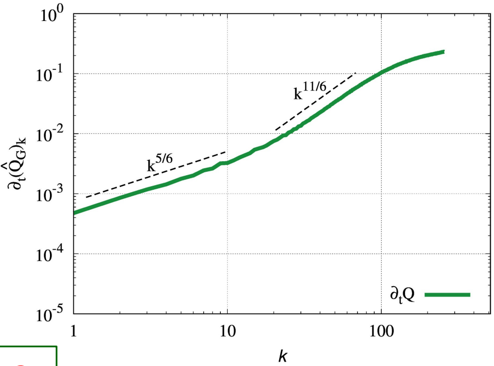

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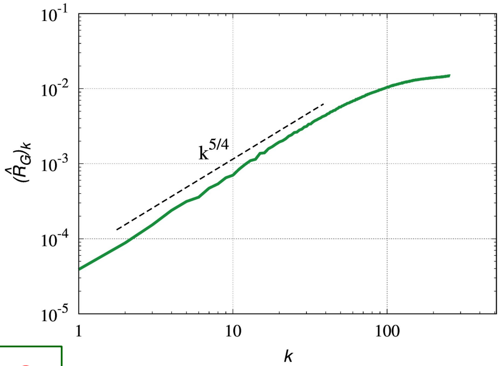

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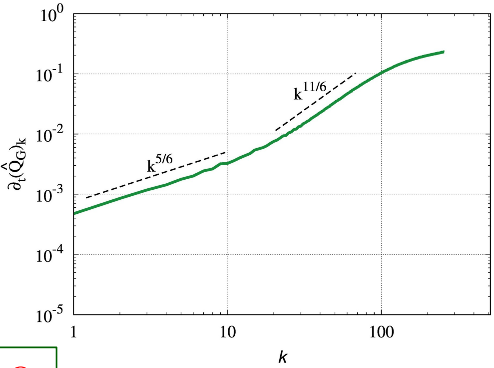

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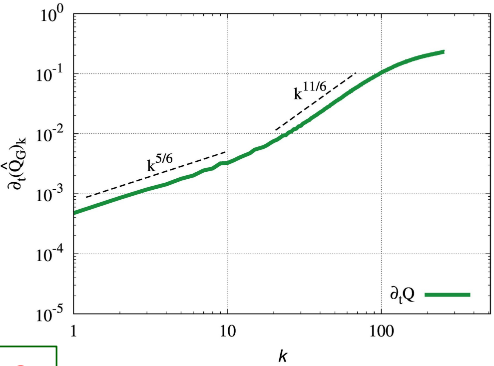

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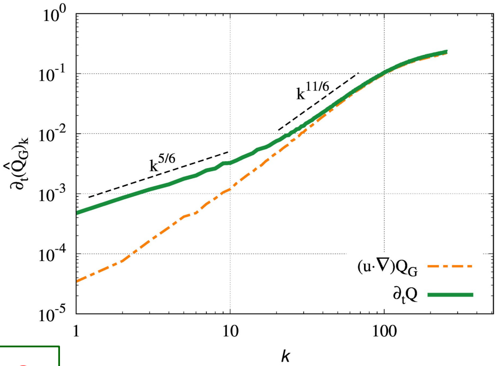

$\mathrm{Re}_\lambda \approx 433$ (1024^3) from https://turbulence.pha.jhu.edu/

$$\boxed{\nabla^2 \tilde{p}^{n+1} = \nabla \cdot \vec{u}^p = 2 \Delta t\, Q_G}$$

Homogeneous isotropic turbulence

New derivations

$$E_k = C_k \varepsilon^{2/3} k^{-5/3}$$

$$P_k = B_p \varepsilon^{4/3} k^{-7/3}$$

$$\hat{p}_k \propto k^{-7/6}$$

$$(\hat{Q}_G)_k \propto k^{-7/6+2} = k^{5/6}$$

$$(\hat{R}_G)_k \propto (k^{5/6})^{3/2} = k^{5/4}$$

$$\boxed{(\hat{r})_k \propto k^{5/6+1} = k^{11/6}}$$

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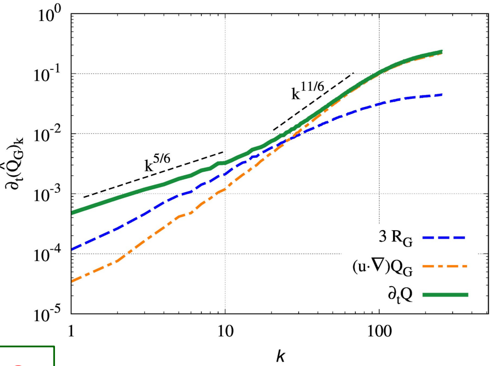

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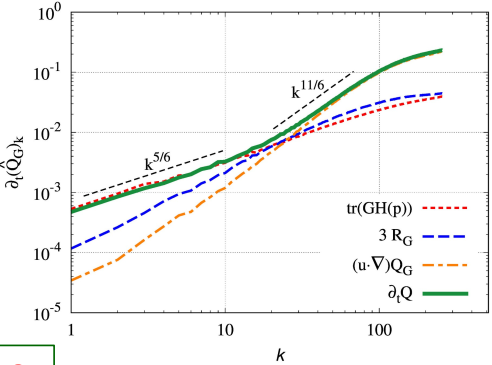

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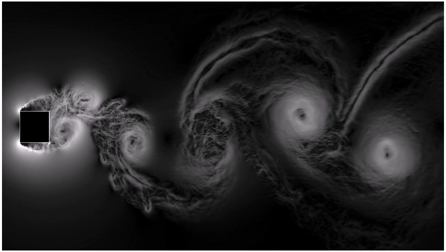
Non-homogeneous turbulent flows

Flow around a square cylinder

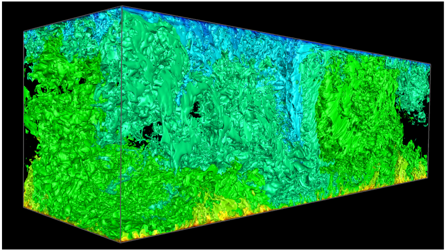


$Re = 22000$ (330M grid points)

$Re = 55000$ (2.6B grid points)

$Re = 100000$ (10B grid points) on-going

Air-filled Rayleigh–Bénard



$Ra = 10^{10}$ (604M grid points)

$Ra = 10^{11}$ (5.7B grid points)

Non-homogeneous turbulent flows

Flow around a square cylinder at $Re = 22000$

$$\widetilde{r}^{o} = \nabla^2 \widetilde{p}^{n} - \nabla \cdot u^{p,n+1} \approx \Delta t \nabla \cdot \frac{\partial u^{p}}{\partial t} \approx 2 \Delta t \frac{\partial Q_G}{\partial t}$$

Non-homogeneous turbulent flows

Flow around a square cylinder at $Re = 22000$

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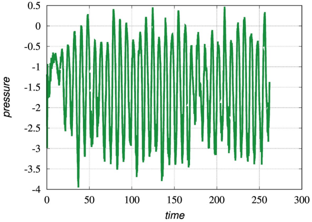
$$\nabla^2 \tilde{p}^n - \nabla^2 \tilde{p}^{n+1} \approx 2 \Delta t \frac{\partial Q_G}{\partial t} \quad \longrightarrow \quad \nabla^2 \frac{\partial p}{\partial t} \approx -2 \frac{\partial Q_G}{\partial t}$$



Non-homogeneous turbulent flows

Flow around a square cylinder at $Re = 22000$

$$\tilde{r}^o = \nabla^2 \tilde{p}^n - \nabla \cdot u^{p,n+1} \approx \Delta t \, \nabla \cdot \frac{\partial u^p}{\partial t} \approx 2 \Delta t \frac{\partial Q_G}{\partial t}$$

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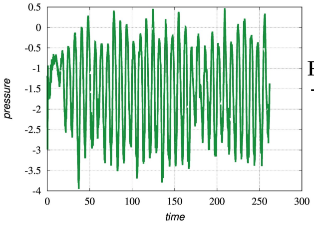
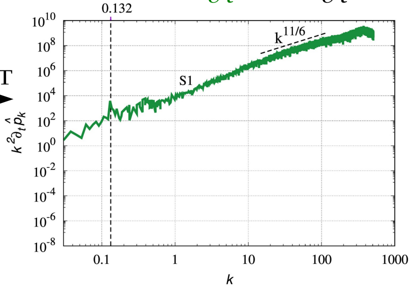


FFT $\longrightarrow$



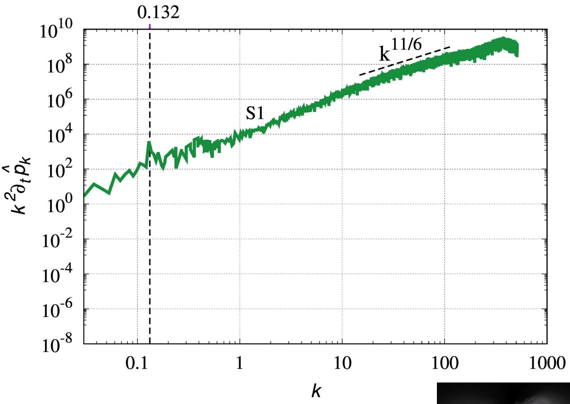
$$k^2 \frac{\partial \hat{p}_k}{\partial t} \approx -2 \frac{\partial \hat{Q}_G}{\partial t}$$

$$\downarrow$$

$$\boxed{k^2 \frac{\partial \hat{p}_k}{\partial t} \propto k^{\beta} \quad \text{with} \quad \beta = 11/6}$$

Non-homogeneous turbulent flows

Flow around a square cylinder at $Re = 22000$



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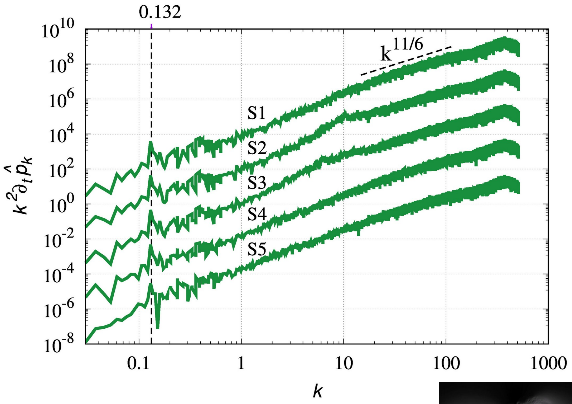
Non-homogeneous turbulent flows

Flow around a square cylinder at $Re = 22000$



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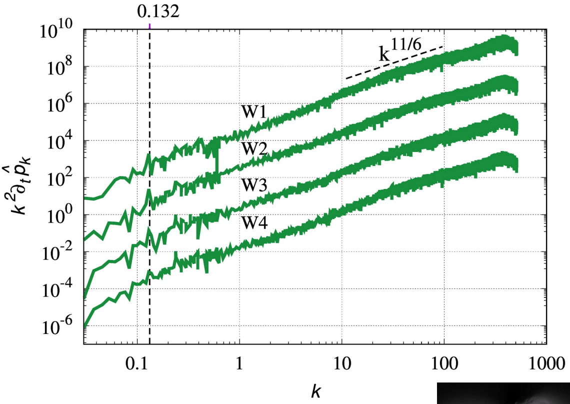
Non-homogeneous turbulent flows

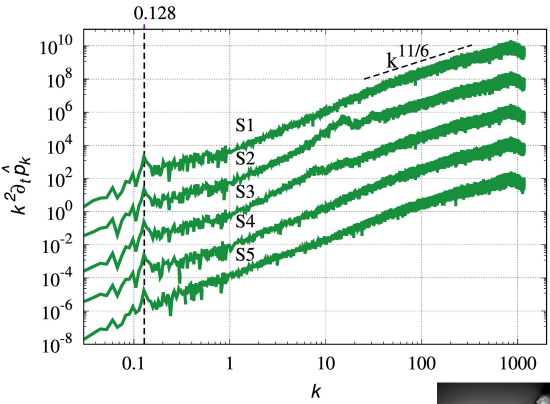
Flow around a square cylinder at $Re = 22000$

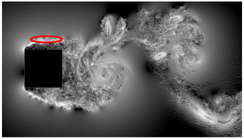


$$\boxed{k^2 \frac{\partial \hat{p}_k}{\partial t} \propto k^{\beta} \quad \text{with} \quad \beta = 11/6}$$

Non-homogeneous turbulent flows

Flow around a square cylinder at $Re = 55000$

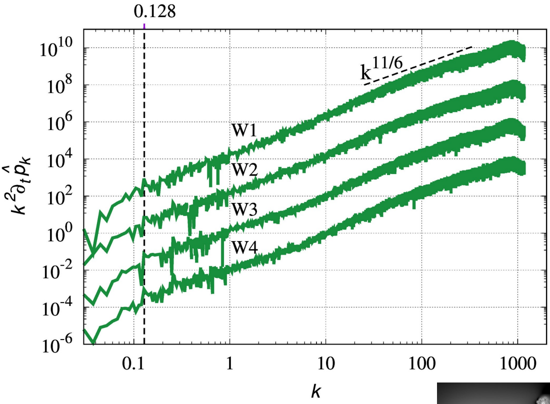


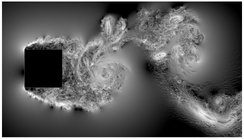


$$\boxed{k^2 \frac{\partial \hat{p}_k}{\partial t} \propto k^{\beta} \quad \text{with} \quad \beta = 11/6}$$

Non-homogeneous turbulent flows

Flow around a square cylinder at $Re = 55000$





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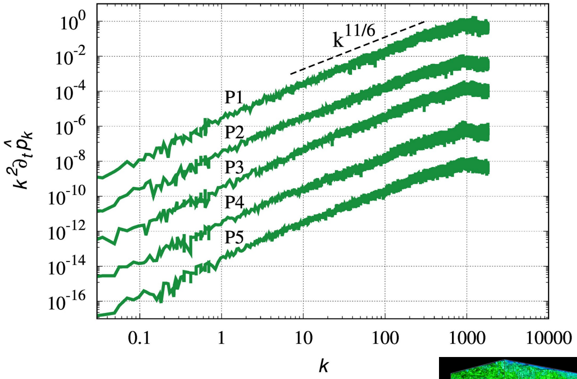
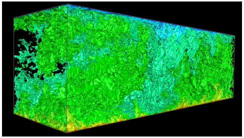
Non-homogeneous turbulent flows

Air-filled Rayleigh–Bénard at $Ra = 10^{10}$

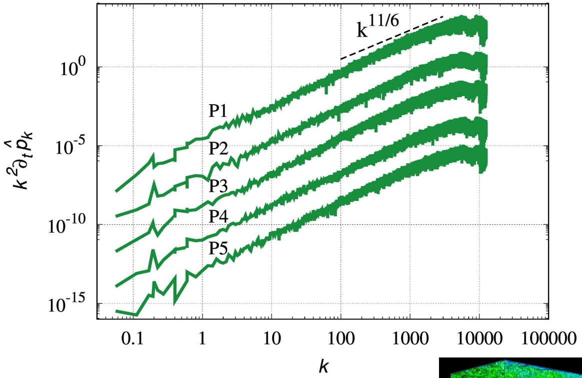




$$k^2 \frac{\partial \hat{p}_k}{\partial t} \propto k^{\beta} \quad \text{with} \quad \beta = 11/6$$

Non-homogeneous turbulent flows

Air-filled Rayleigh–Bénard at $Ra = 10^{11}$



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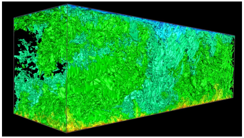
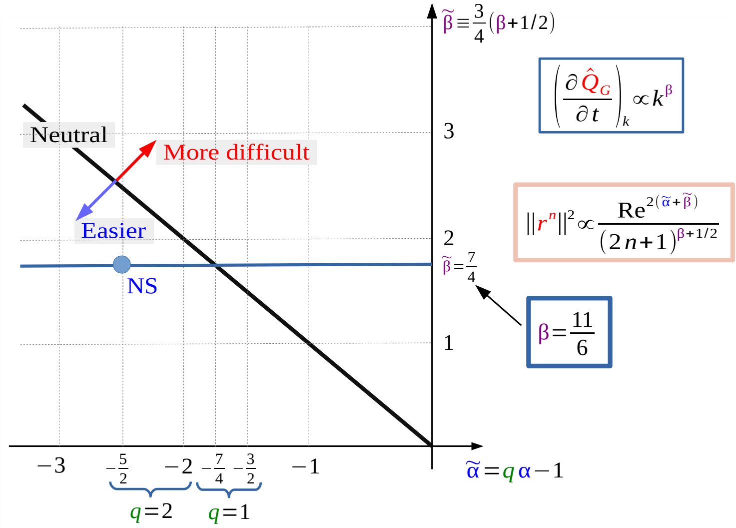


Solver convergence

$\{\tilde{\alpha}, \tilde{\beta}\}$ phase space



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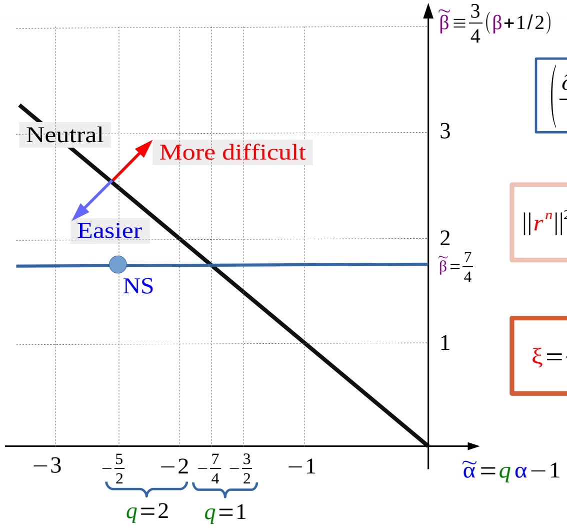


$$\boxed{\left(\frac{\partial \hat{Q}_G}{\partial t}\right)_k \propto k^{\beta}}$$

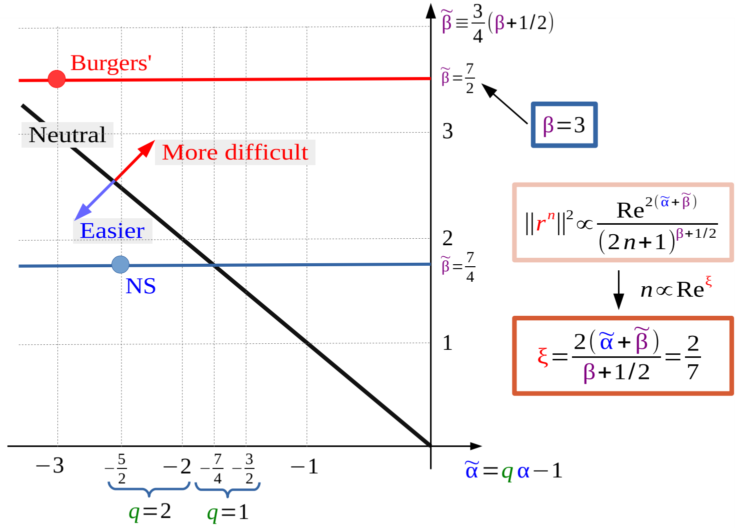
$$\boxed{\|r^n\|^2 \propto \frac{\mathrm{Re}^{2(\tilde{\alpha}+\tilde{\beta})}}{(2n+1)^{\beta+1/2}}}$$

$$\downarrow \quad n \propto \mathrm{Re}^{\xi}$$

$$\boxed{\xi = \frac{2(\tilde{\alpha}+\tilde{\beta})}{\beta+1/2} = -\frac{9}{7}}$$

Solver convergence for Burgers' equation

$\{\tilde{\alpha}, \tilde{\beta}\}$ phase space



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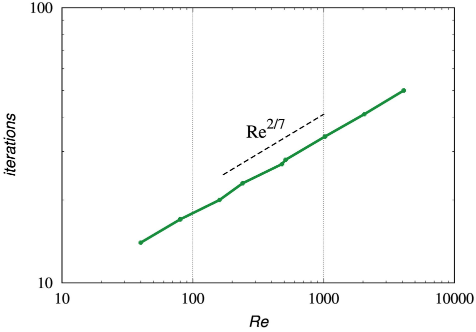

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Concluding remarks

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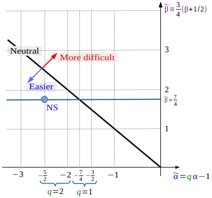

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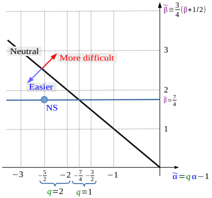
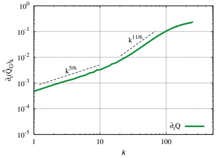
  - For HIT ✓





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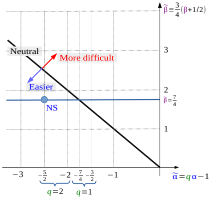

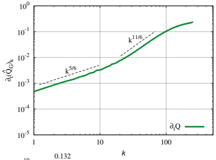

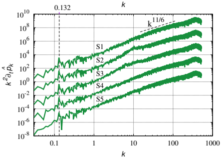

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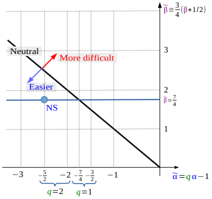
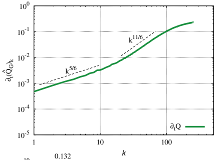
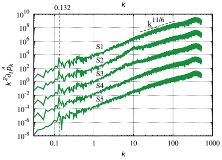
On-going research:

- Numerical analysis of the scaling of number of solver iterations with *Re*







Thank you for your attendance