



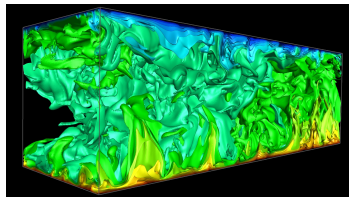
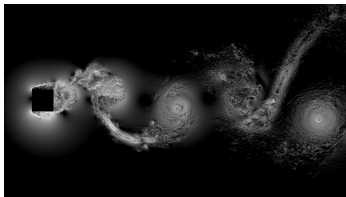
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UNIVERSITAT POLITÈCNICA DE CATALUNYA

Conservative methods for multi-physics flows

F.Xavier Trias

Heat and Mass Transfer Technological Center, Technical University of Catalonia (UPC)

NUM-SCARS Workshop, CWI Amsterdam, 22-23 January 2026





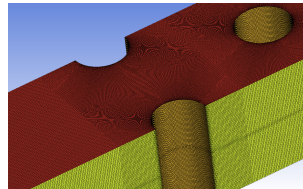
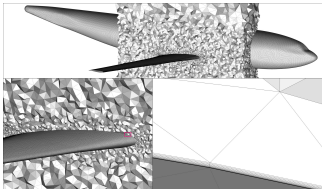
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Conservative methods for multi-physics flows on complex geometries

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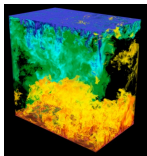


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- 1 Motivation
- 2 Preserving symmetries at discrete level
- 3 Energy exchange in multiphysics
- 4 Take-away messages

General motivation: (very) large-scale DNS/LES

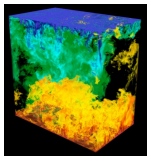
For canonical academic flows...



DNS {

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MareNostrum 5-ACC
(Barcelona)



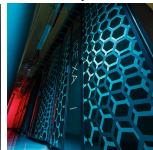
rank #8
1120 nodes with:
2x Intel Shappire Rapids 8460
1x NVIDIA Hopper 64 GB HBM
1x Infiniband NDR200

Snellius
(Amsterdam)



rank #165
714 nodes with:
2x AMD EPYC 9654 96C
1x Infiniband NDR200

TSUBAME4.0
(Tokyo)



rank #31
240 nodes with:
2x AMD EPYC 9654 96-Core
4x NVIDIA H100 SXM5
1x Infiniband NDR200

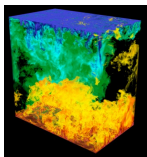


DNS

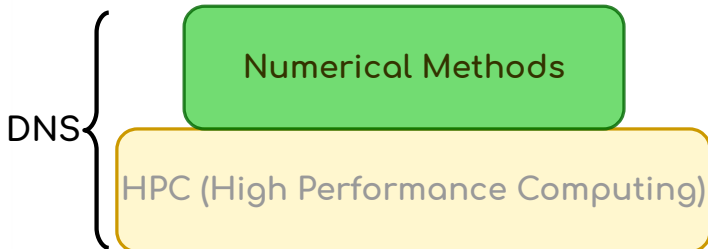
HPC (High Performance Computing)

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For canonical academic flows...



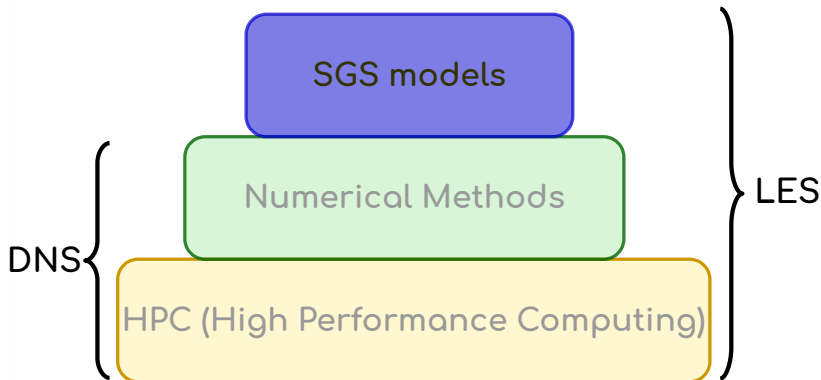
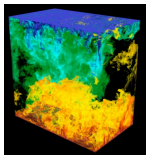
How to properly discretize NS?



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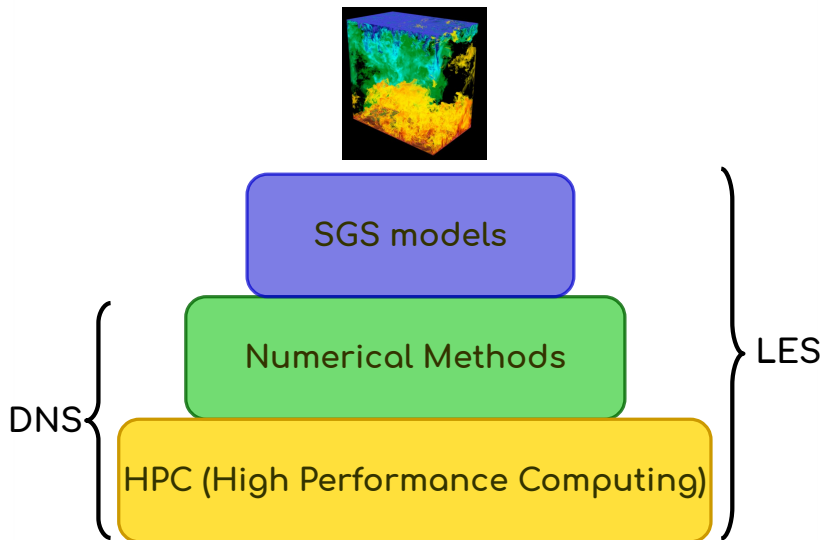
For canonical academic flows...

How to
properly
model SGS?



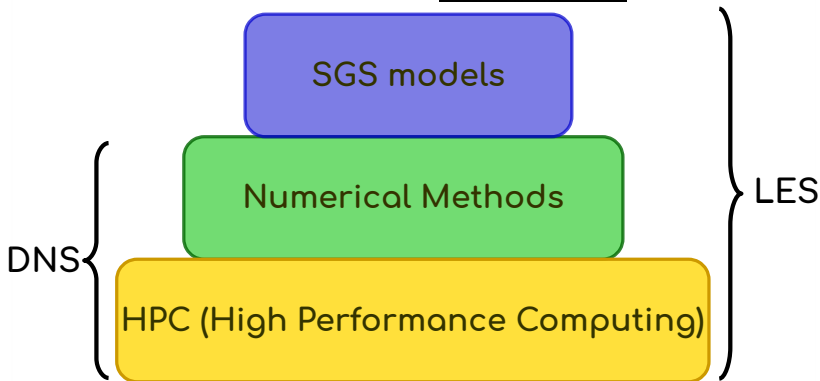
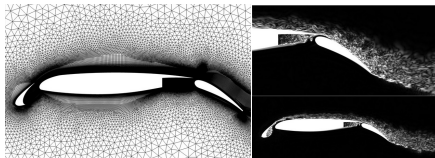
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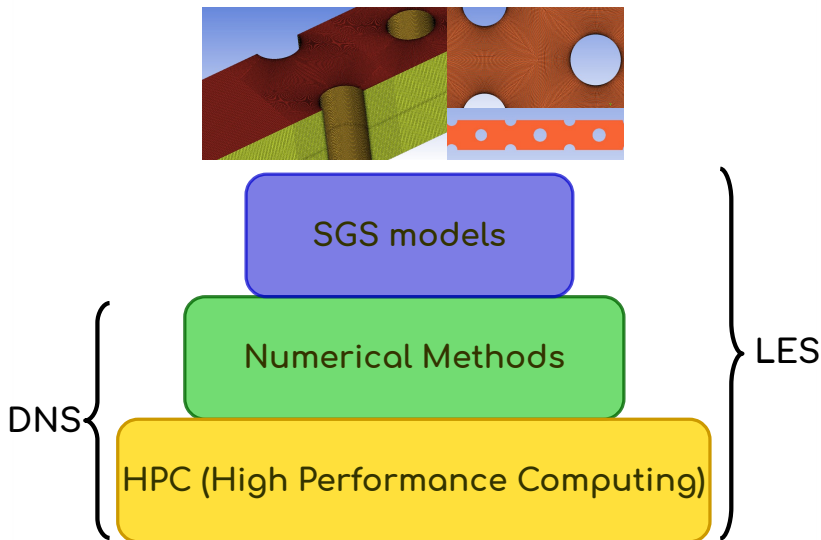
General motivation: (very) large-scale DNS/LES

...but also for industrial applications



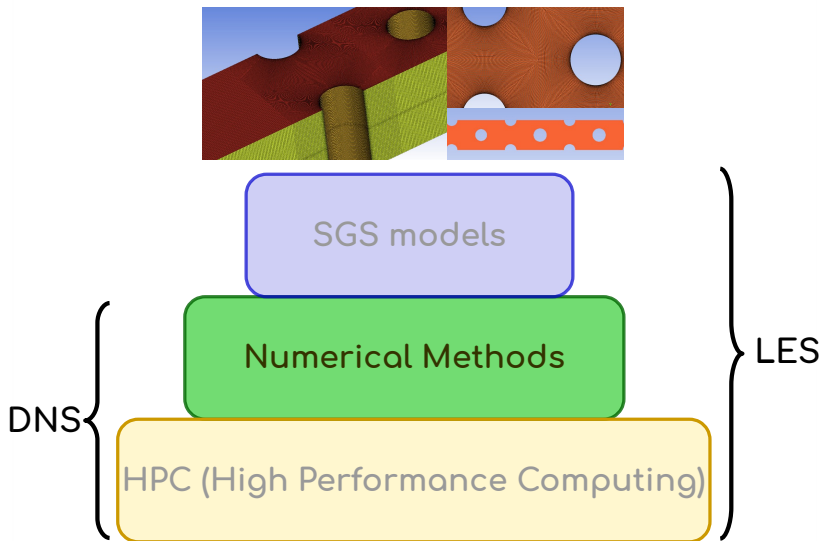
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Numerical methods for DNS/LES

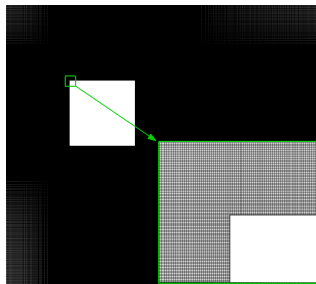
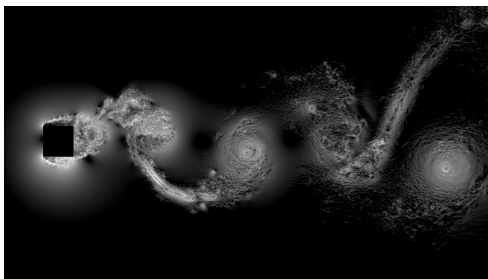
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Numerical methods for DNS/LES

Research question #1:

- Can we construct numerical discretizations of the Navier-Stokes equations suitable for **complex geometries**, such that the **symmetry properties** are exactly preserved?



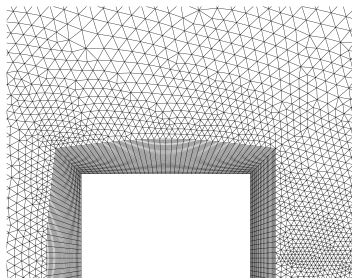
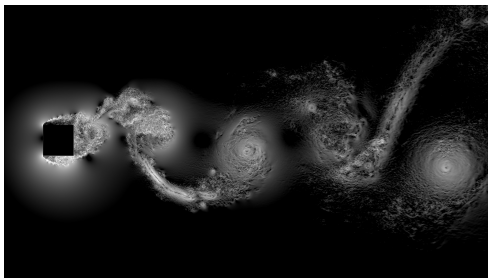
DNS¹ of the flow around a square cylinder at $Re = 55000$ (2.6B grid points)

¹F.X.Trias, À.Alsaltí, A.Oliva. *On the Reynolds-number scaling of Poisson solver complexity*, [arXiv: 2512.22644](https://arxiv.org/abs/2512.22644), December 2025.

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Motivation

Frequently used general purpose CFD codes:

- STAR-CCM+



SIEMENS

- ANSYS-FLUENT



- Code-Saturne



- OpenFOAM



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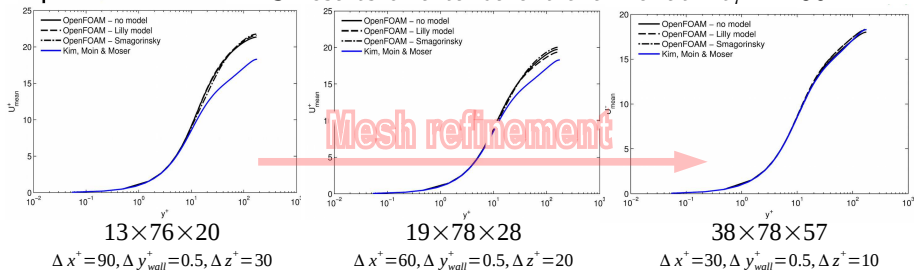


Main common characteristics of LES in such codes:

- **Unstructured finite volume** method, **collocated** grid
- Second-order spatial and temporal discretisation
- Eddy-viscosity type LES models

Motivation

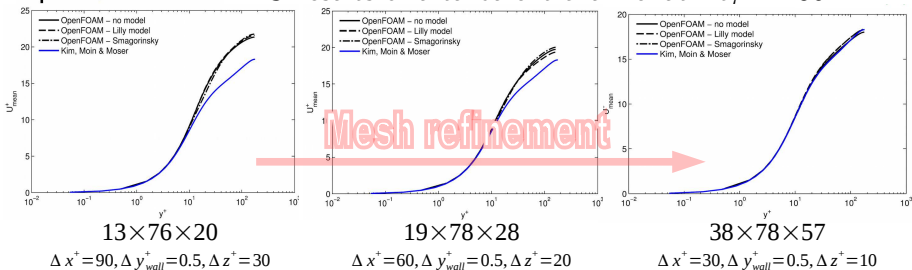
OpenFOAM® LES³ results of a turbulent channel at $Re_\tau = 180$



³E.M.J.Komen, L.H.Camilo, A.Shams, B.J.Geurts, B.Koren. *A quantification method for numerical dissipation in quasi-DNS and under-resolved DNS, and effects of numerical dissipation in quasi-DNS and under-resolved DNS of turbulent channel flows*, **Journal of Computational Physics**, 345, 565-595, 2017.

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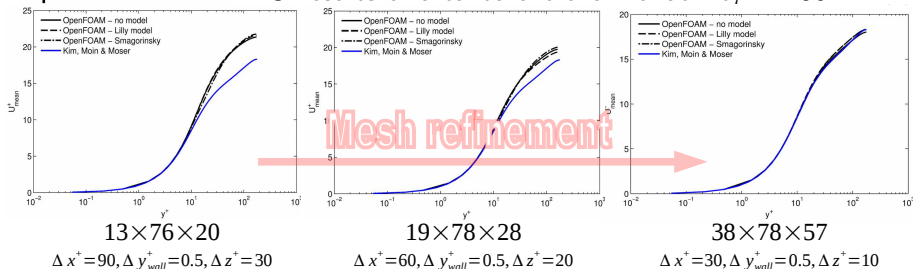


- Are LES results are merit of the SGS model?

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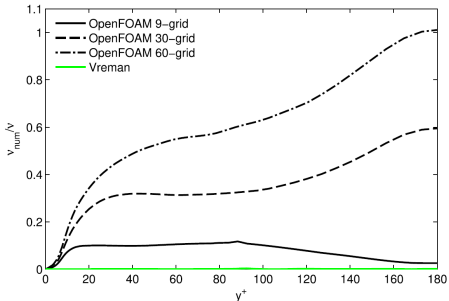
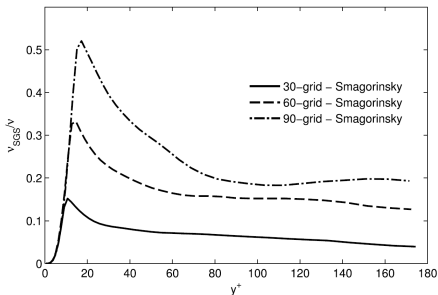


- Are LES results are merit of the SGS model? Apparently **NOT!!!** ❌

³E.M.J.Komen, L.H.Camilo, A.Shams, B.J.Geurts, B.Koren. *A quantification method for numerical dissipation in quasi-DNS and under-resolved DNS, and effects of numerical dissipation in quasi-DNS and under-resolved DNS of turbulent channel flows*, **Journal of Computational Physics**, 345, 565-595, 2017.

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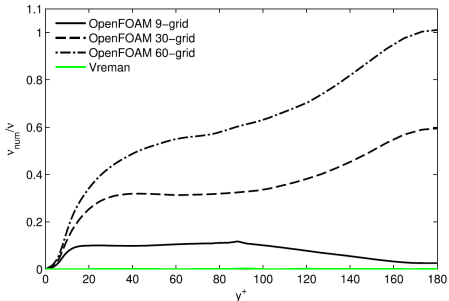
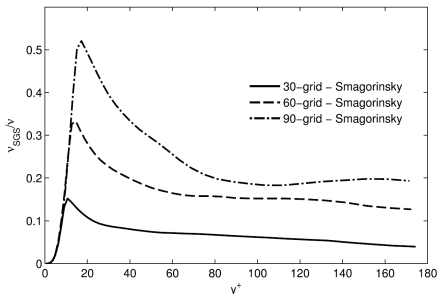


$$\nu_{num} \neq 0$$

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Open ∇ FOAM® LES⁴ results of a turbulent channel at $Re_\tau = 180$



$$\nu_{SGS} < \nu_{num} \neq 0$$

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Symmetry-preserving discretization

Continuous

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{C}(\mathbf{u}, \mathbf{u}) = \nu \nabla^2 \mathbf{u} - \nabla p$$

$$\nabla \cdot \mathbf{u} = 0$$

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Discrete

$$\Omega \frac{d\mathbf{u}_h}{dt} + \mathbf{C}(\mathbf{u}_h) \mathbf{u}_h = \mathbf{D} \mathbf{u}_h - \mathbf{G} \mathbf{p}_h$$

$$\mathbf{M} \mathbf{u}_h = \mathbf{0}_h$$

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$$\langle \mathbf{C}(\mathbf{u}, \varphi_1), \varphi_2 \rangle = - \langle \mathbf{C}(\mathbf{u}, \varphi_2), \varphi_1 \rangle$$

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$$\mathbf{D} = \mathbf{D}^T \quad \text{def --}$$

Why collocated arrangements are so popular?

- STAR-CCM+



SIEMENS



- ANSYS-FLUENT



- Code-Saturne



- OpenFOAM

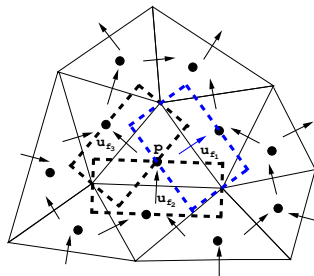
OpenFOAM®



$$\Omega_s \frac{d\mathbf{u}_s}{dt} + \mathbf{C}(\mathbf{u}_s) \mathbf{u}_s = \mathbf{D} \mathbf{u}_s - \mathbf{G} \mathbf{p}_c; \quad \mathbf{M} \mathbf{u}_s = \mathbf{0}_c$$

In staggered meshes

- p - $\mathbf{u}_s$ coupling is naturally solved ✓
- $\mathbf{C}(\mathbf{u}_s)$ and $\mathbf{D}$ difficult to discretize ✗



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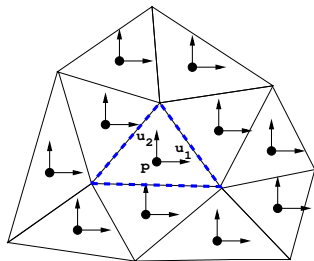
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In collocated meshes

- $\mathbf{p}-\mathbf{u}_c$ coupling is cumbersome ✗
- $\mathbf{C}(\mathbf{u}_s)$ and $\mathbf{D}$ easy to discretize ✓
- Cheaper, less memory, ... ✓



Why colocated arrangements are so popular?

Everything is easy except the pressure-velocity coupling...

- STAR-CCM+



SIEMENS



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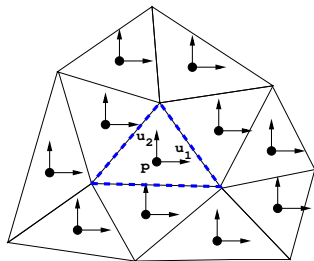
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Pressure-velocity coupling on collocated grids

A vicious circle that cannot be broken...

In summary⁵:

- Mass: $M\Gamma_{c \rightarrow s} u_c = M\Gamma_{c \rightarrow s} u_c - L_c L^{-1} M\Gamma_{c \rightarrow s} u_c \approx 0_c \quad \times$
- Energy: $p_c (L - L_c) p_c \neq 0 \quad \times$

5F.X.Trias, O.Lehmkuhl, A.Oliva, C.D.Pérez-Segarra, R.W.C.P.Verstappen. Symmetry-preserving discretization of Navier-Stokes equations on collocated unstructured grids, *Journal of Computational Physics*, 258 (1): 246-267, 2014.

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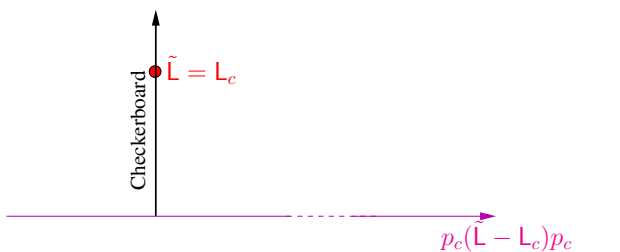
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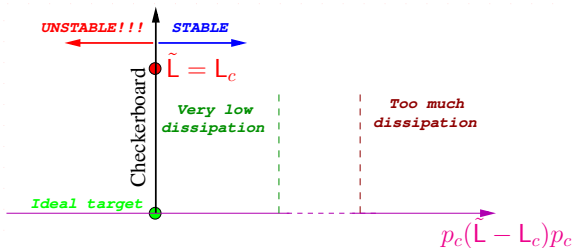
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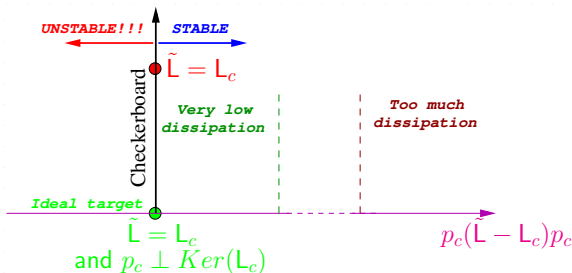
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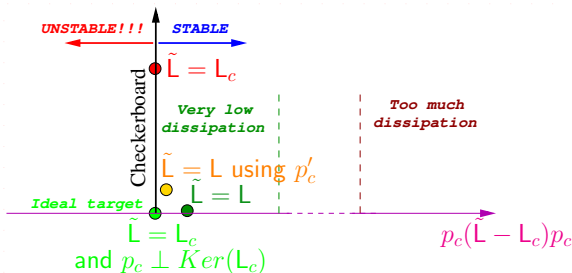
⁵Shashank, J.Larsson, G.laccarino. *A co-located incompressible Navier-Stokes solver with exact mass, momentum and kinetic energy conservation in the inviscid limit*, *Journal of Computational Physics*, 229: 4425-4430,2010.

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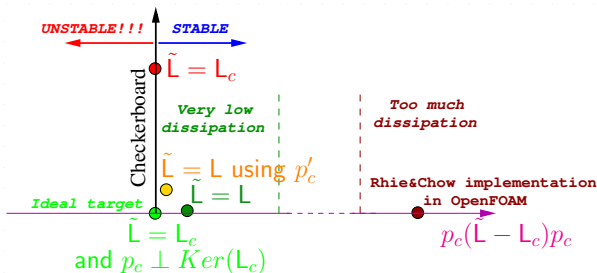
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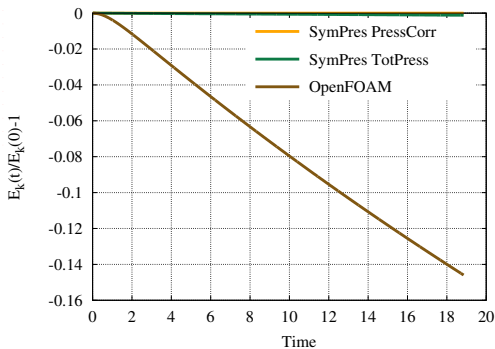
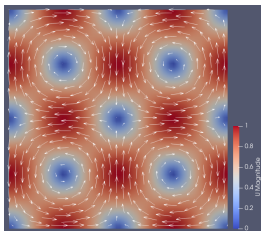
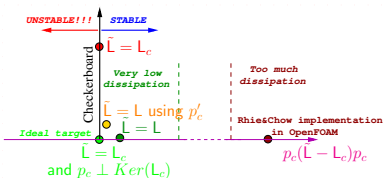
- Mass: $M\Gamma_{c \rightarrow s} u_c = M\Gamma_{c \rightarrow s} u_c - L_c L^{-1} M\Gamma_{c \rightarrow s} u_c \approx 0_c \quad \text{X}$
- Energy: $p_c (L - L_c) p_c \neq 0 \quad \text{X}$



⁵E.Komen, J.A.Hopman, E.M.A.Frederix, F.X.Trias, R.W.C.P.Verstappen. A symmetry-preserving second-order time-accurate PISO-based method, **Computers & Fluids**, 225:104979, 2021.

Pressure-velocity coupling on collocated grids

A vicious circle that ~~cannot be broken~~ can almost be broken...

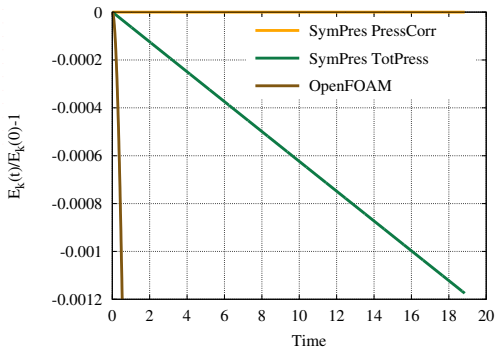
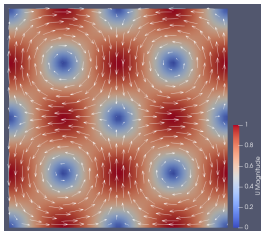
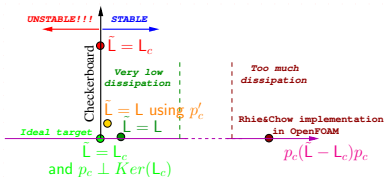


Results for an inviscid Taylor-Green vortex⁶

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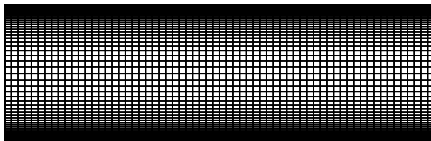


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Pressure-velocity coupling on collocated grids

... and make it **unconditionally stable**⁷ even on extremely bad meshes!



On a symmetry-preserving unconditionally stable projection method on collocated unstructured grids for incompressible flows

D. Santos¹, J.A. Hopman, C.D. Pérez-Segarra, F.X. Trias

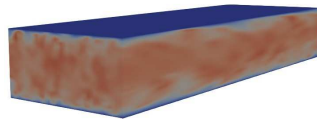
Received 10 May 2024; revised 10 May 2024; accepted 10 May 2024; available online 10 May 2024

ARTICLE INFO

Keywords:
Symmetry-preserving
Projection method
PISO
Unconditionally stable
Energy-preserving
PVM

ABSTRACT

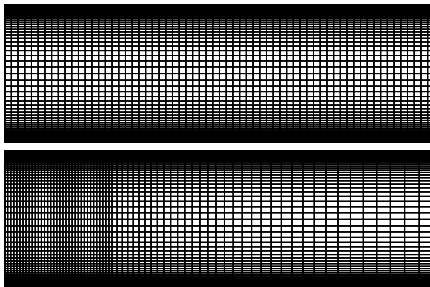
This work establishes the conditions for having a symmetry-preserving unconditionally stable projection method for incompressible flows, such as PISO or Fractional Step Method (FSM), on collocated unstructured grids based on a compact stencil Poisson equation. Its formulation is based on preserving the underlying symmetries of the differential operators. In addition, a general theorem for these types of projection methods will be proven. To establish an unconditionally stable method, the chosen grid, the mechanical requirements for the operator and the geometrical conditions are mathematically verified, even in cases where the mesh is highly distorted. This will be proven both theoretically and numerically. One criterion of robust kinetic energy is



⁷D.Santos, J.A.Hopman, C.D.Pérez-Segarra, F.X.Trias. *On a symmetry-preserving unconditionally stable projection method on collocated unstructured grids for incompressible flows*, **Journal of Computational Physics**, 523:113631, 2025.

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On a symmetry-preserving unconditionally stable projection method on collocated unstructured grids for incompressible flows

D. Santos^a, J.A. Hopman, C.D. Pérez-Segarra, F.X. Trias

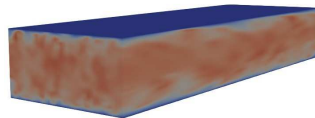
^a Institut de Recerca en Energia del Baix Llobregat (IREB), Departament d'Enginyeria i Ciències de l'Espai, Universitat de Vic, 08500 Vic, Spain

ARTICLE INFO

Keywords:
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Projection method
PISO
Unconditionally stable
Energy-preserving
PIM

ABSTRACT

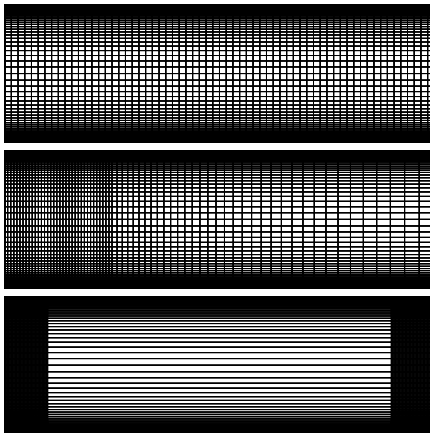
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⁷D.Santos, J.A.Hopman, C.D.Pérez-Segarra, F.X.Trias. *On a symmetry-preserving unconditionally stable projection method on collocated unstructured grids for incompressible flows*, **Journal of Computational Physics**, 523:113631, 2025.

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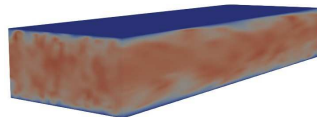
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Projection method
Unstructured grids
Incompressible flows

ABSTRACT

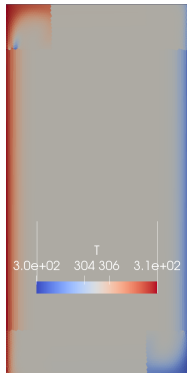
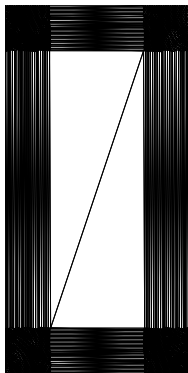
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⁷D.Santos, J.A.Hopman, C.D.Pérez-Segarra, F.X.Trias. *On a symmetry-preserving unconditionally stable projection method on collocated unstructured grids for incompressible flows*, **Journal of Computational Physics**, 523:111631, 2025.

Pressure-velocity coupling on collocated grids

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Journal of Computational Physics

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D. Santos¹, J.A. Hopman, C.D. Pérez-Segarra, F.X. Trias

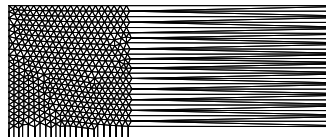
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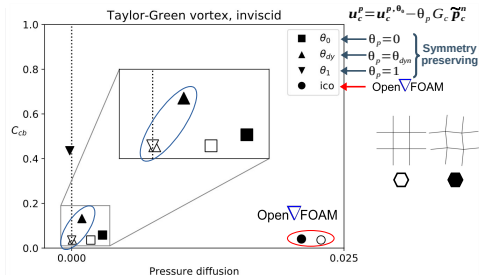
This work establishes the conditions for having a symmetry-preserving unconditionally stable projection method for incompressible flows, such as PISO or Fractional Step Method (FSM), on collocated unstructured grids based on a compact linear Poisson equation. Its formulation is based on projecting the unphysical components of the differential operators. In addition, a general theorem for these types of projection methods will be proven. Its viability as an unconditionally stable method, the theorem gives the mathematical requirements for the operators and the geometrical conditions a mesh must satisfy, even in cases where the mesh is highly distorted. This will be proven both theoretically and numerically. Characteristics of robust kinetic energy



⁷D.Santos, J.A.Hopman, C.D.Pérez-Segarra, F.X.Trias. *On a symmetry-preserving unconditionally stable projection method on collocated unstructured grids for incompressible flows*, **Journal of Computational Physics**, 523:113631, 2025.

Pressure-velocity coupling on collocated grids

... and minimize the checkerboard⁸ problem with a minimal artificial dissipation!



Quantifying the checkerboard problem to reduce numerical dissipation

J. A. Hopman^{1,2}, D. Santos¹, À. Alsalti-Baldellou^{1,2}, J. Rigola¹, F. X. Trias¹

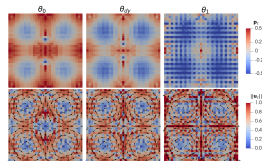
¹ IREC 2018-2021 TITELUP-TECHNOLOGICAL CENTER OF RESEARCH, UNIVERSITAT DE VALÈNCIA, 46100 BURJASSOT, SPAIN
² TITELUP-TECHNOLOGICAL CENTER OF RESEARCH, UNIVERSITAT DE VALÈNCIA, 46100 BURJASSOT, SPAIN

ARTICLE INFO

Keywords:
checkerboard
dissipation
numerical error

ABSTRACT

This work presents a comprehensive implementation of various methods to solving the checkerboard problem using a projection method, and their relation to the occurrence and management of checkerboard dissipation. It employs an algebraic symmetry-preserving approach, clarifying the detection and implementation of its own operation while also addressing the numerical error. The lack of a proper definition for the checkerboard problem is addressed by preparing a ghost-based coefficient. The coefficient, mixed in the dissipation between the complex and multi-cell Laplacian operation, while in a case of multi-cell sub-grid scale, with a sub-grid-based, global, normalized, non-dimensional value. The influence of mesh and time-step



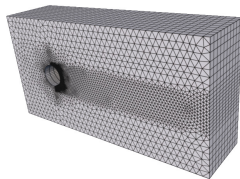
⁸J.A.Hopman, D.Santos, À.Alsalti, J.Rigola, F.X.Trias. *Quantifying the checkerboard problem to reduce numerical dissipation*, **Journal of Computational Physics**, 521:111337, 2025. [arXiv: 2408.06821](https://arxiv.org/abs/2408.06821)

Pressure-velocity coupling on collocated grids

Summary

A robust and reliable symmetry-preserving collocated formulation...

- Virtually **artificial-dissipation-free**
- Suitable for **unstructured** grids

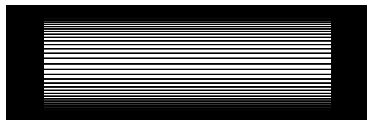


Pressure-velocity coupling on collocated grids

Summary

A robust and reliable symmetry-preserving collocated formulation...

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- **Unconditionally stable** even on:
 - Very low quality grids

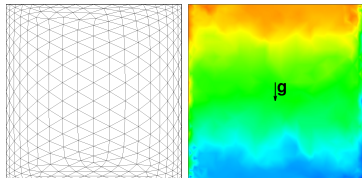


Pressure-velocity coupling on collocated grids

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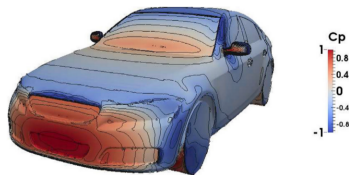
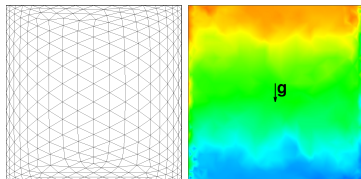


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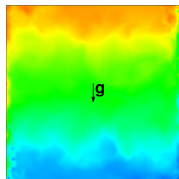
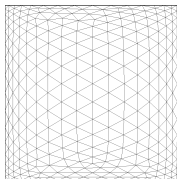


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 - Very coarse grids
- Suitable for **industrial** applications
- Also available in **OpenFOAM**



Open▽FOAM

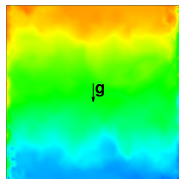
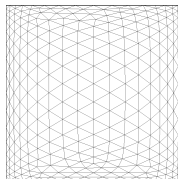


Pressure-velocity coupling on collocated grids

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Open▽FOAM

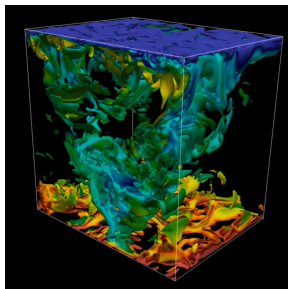


Research question #2:

- What happens to the **other forms of energy** and their **exchanges**?

Buoyancy-driven flows

Continuous formulation



$$E_p$$

$$-(y, T)$$

$$E_k$$

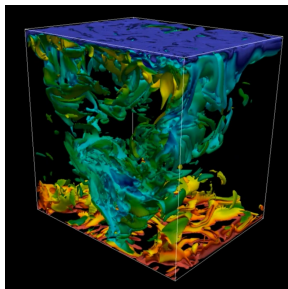
$$\frac{1}{2}(\mathbf{u}, \mathbf{u})$$

$$E_i$$

$$\int_{\Omega} T d\Omega$$

Buoyancy-driven flows

Continuous formulation



$$E_p$$
$$-(y, T)$$

$$E_k$$
$$\frac{1}{2}(\mathbf{u}, \mathbf{u})$$

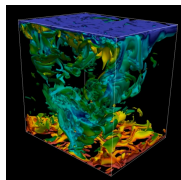
$$E_i$$
$$\int_{\Omega} T d\Omega$$
$$\sqrt{\frac{1}{\text{Pr Ra}}} Nu_c$$
$$\sqrt{\frac{1}{\text{Pr Ra}}} Nu_h$$

Buoyancy-driven flows

Continuous formulation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + \underbrace{T \mathbf{e}_y}_{\text{Boussinesq}} \quad \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T$$



$$E_p$$

$$-(y, T)$$

$$E_k$$

$$\frac{1}{2}(\mathbf{u}, \mathbf{u})$$

$$E_i$$

$$\int_{\Omega} T d\Omega$$

$$\sqrt{\frac{1}{\text{Pr Ra}}} Nu_c$$

$$\sqrt{\frac{1}{\text{Pr Ra}}} Nu_h$$

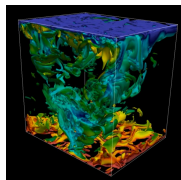
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$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T$$

$$\frac{dE_p}{dt} = (y, \nabla \cdot (\mathbf{u} T)) - \sqrt{\frac{1}{\text{Pr Ra}}} (y, \nabla^2 T)$$



$$E_p$$

$$-(y, T)$$

$$E_k$$

$$\frac{1}{2}(\mathbf{u}, \mathbf{u})$$

$$E_i$$

$$\int_{\Omega} T d\Omega$$

$$\sqrt{\frac{1}{\text{Pr Ra}}} Nu_c$$

$$\sqrt{\frac{1}{\text{Pr Ra}}} Nu_h$$

Buoyancy-driven flows

Continuous formulation

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + \underbrace{T \mathbf{e}_y}_{\text{Boussinesq}} \quad \nabla \cdot \mathbf{u} = 0 \\ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T &= \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T \end{aligned}$$

$$\Phi = \|\nabla \mathbf{u}\|^2 \varepsilon_U = \int_{\Omega} \Phi d\Omega$$

$$(\mathbf{v}, T) = \int_{\Omega} \mathbf{v} T d\Omega$$

$$\frac{dE_p}{dt} = (y, \nabla \cdot (\mathbf{u} T)) - \sqrt{\frac{1}{\text{Pr Ra}}} (y, \nabla^2 T)$$

$$\frac{dE_k}{dt} = -\sqrt{\frac{\text{Pr}}{\text{Ra}}} \varepsilon_U + (\mathbf{v}, T)$$

$$\begin{array}{c} E_p \\ -(y, T) \end{array}$$

$$\begin{array}{c} E_k \\ \frac{1}{2}(\mathbf{u}, \mathbf{u}) \end{array}$$

$$\begin{array}{c} \sqrt{\frac{1}{\text{Pr Ra}}} Nu_c \\ E_i \\ \int_{\Omega} T d\Omega \\ \sqrt{\frac{1}{\text{Pr Ra}}} Nu_h \end{array}$$

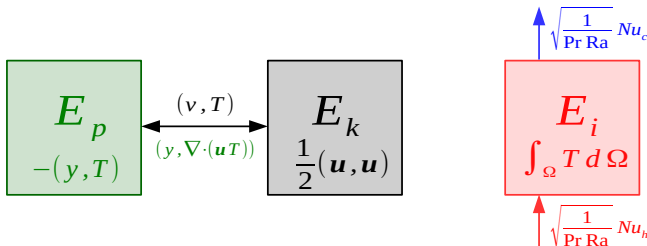
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$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + \underbrace{T \mathbf{e}_y}_{\text{Boussinesq}} & \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T &= \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T & \Phi &= \|\nabla \mathbf{u}\|^2 \varepsilon_U = \int_{\Omega} \Phi d\Omega \\ & & (v, T) &= \int_{\Omega} v T d\Omega\end{aligned}$$

$$\frac{dE_p}{dt} = (y, \nabla \cdot (\mathbf{u} T)) - \sqrt{\frac{1}{\text{Pr Ra}}} (y, \nabla^2 T)$$

$$\frac{dE_k}{dt} = -\sqrt{\frac{\text{Pr}}{\text{Ra}}} \varepsilon_U + (v, T)$$



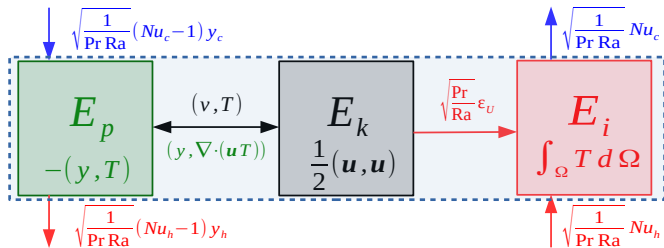
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$$\frac{dE_k}{dt} = -\sqrt{\frac{\text{Pr}}{\text{Ra}}} \varepsilon_U + (v, T)$$



Buoyancy-driven flows

Discrete formulation on **staggered** grids: viscous dissipation⁹

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + \underbrace{T \mathbf{e}_y}_{\text{Boussinesq}} \quad \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T + \text{Ge} \sqrt{\frac{\text{Pr}}{\text{Ra}}} \Phi$$

NOTE: for many applications

$$\text{Ge} \ll 1$$

expect for:

→ For large values of H
e.g. Geophysical flows

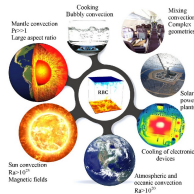
→ For large values of g
e.g. high rotational devices

$$\text{Ra} = \frac{g \beta \Delta T H^3}{\nu \alpha}$$

$$\text{Pr} = \frac{\nu}{\alpha}$$

$$\text{Ge} = \frac{\beta g H}{c_v}$$

Gebhart number



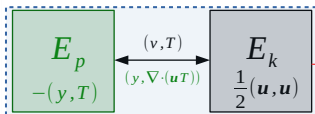
⁹B.Sanderse, F.X.Trias. *Energy-consistent discretization of viscous dissipation with application to natural convection flow*, **Computers & Fluids**, 286:106473, 2025. [arXiv:2307.10874](https://arxiv.org/abs/2307.10874)

Buoyancy-driven flows

Discrete formulation on **staggered** grids: buoyancy term

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + \underbrace{T \mathbf{e}_y}_{\text{Boussinesq}} \quad \nabla \cdot \mathbf{u} = 0 \\ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T &= \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T + \underbrace{\text{Ge}}_{\text{Boussinesq}} \sqrt{\frac{\text{Pr}}{\text{Ra}}} \Phi \quad \Phi = \|\nabla \mathbf{u}\|^2 \varepsilon_U = \int_{\Omega} \Phi d\Omega \\ &\quad (v, T) = \int_{\Omega} v T d\Omega \end{aligned}$$

$$\underbrace{T \mathbf{e}_y}_{\text{Boussinesq}} \quad (\mathbf{u}, \underbrace{T \mathbf{e}_y}_{\text{Boussinesq}}) = (v, T)$$



Buoyancy-driven flows

Discrete formulation on **staggered** grids: buoyancy term

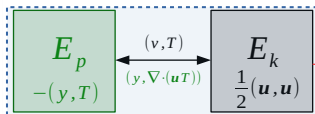
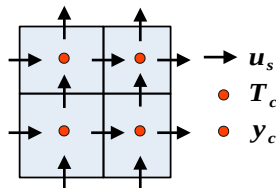
$$\begin{aligned}
 \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + T \mathbf{e}_y \quad \text{Boussinesq} \quad \nabla \cdot \mathbf{u} = 0 \\
 \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T &= \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T + \text{Ge} \sqrt{\frac{\text{Pr}}{\text{Ra}}} \Phi \quad \Phi = \|\nabla \mathbf{u}\|^2 \varepsilon_U = \int_{\Omega} \Phi d\Omega \\
 &\quad (v, T) = \int_{\Omega} v T d\Omega
 \end{aligned}$$

$$T \mathbf{e}_y$$

$$(\mathbf{u}, T \mathbf{e}_y) = (v, T)$$

$$(\mathbf{u} \cdot \nabla) T$$

$$(y, (\mathbf{u} \cdot \nabla) T) = -(v, T)$$



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$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + \underbrace{T \mathbf{e}_y}_{\text{Boussinesq}} \quad \nabla \cdot \mathbf{u} = 0$$

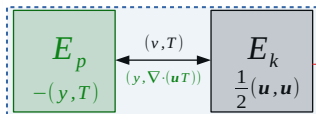
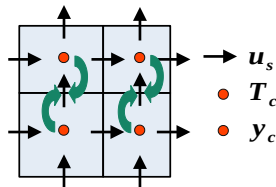
$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T + \text{Ge} \sqrt{\frac{\text{Pr}}{\text{Ra}}} \Phi \quad \Phi = \|\nabla \mathbf{u}\|^2 \varepsilon_U = \int_{\Omega} \Phi d\Omega$$

$$(v, T) = \int_{\Omega} v T d\Omega$$

$$\underbrace{T \mathbf{e}_y}_{E_y \tilde{\Pi}_{cs} T_c} \quad (\mathbf{u}, \underbrace{T \mathbf{e}_y}) = (v, T)$$

$$\mathbf{u}_s \cdot \Omega_s E_y \tilde{\Pi}_{cs} T_c = \mathbf{u}_s^y \Omega_s \tilde{\Pi}_{cs} T_c$$

$$\underbrace{(\mathbf{u} \cdot \nabla) T}_{(y, \underbrace{(\mathbf{u} \cdot \nabla) T})} = -(v, T)$$



Buoyancy-driven flows

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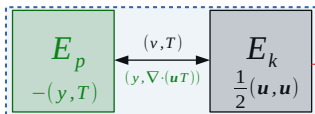
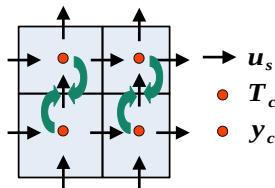
$$\underbrace{T \mathbf{e}_y}_{E_y \tilde{\Pi}_{cs} T_c} \quad (\mathbf{u}, \underbrace{T \mathbf{e}_y}) = (v, T)$$

$$u_s \cdot \Omega_s E_y \tilde{\Pi}_{cs} T_c = u_s^y \Omega_s \tilde{\Pi}_{cs} T_c$$

$$\underbrace{(\mathbf{u} \cdot \nabla) T}_{M U_s \Pi_{cs} T_c} \quad (y, \underbrace{(\mathbf{u} \cdot \nabla) T}) = -(v, T)$$

$$y_c \cdot M U_s \Pi_{cs} T_c = -y_c \cdot G^T \Omega_s U_s \Pi_{cs} T_c$$

$$= -u_s^y \Omega_s \Pi_{cs} T_c$$



Buoyancy-driven flows

Discrete formulation on **staggered** grids: buoyancy term

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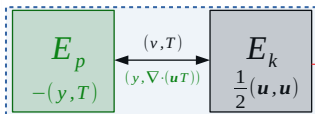
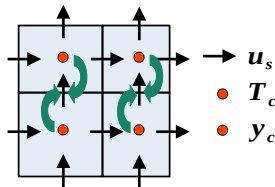
$$\underbrace{T \mathbf{e}_y}_{E_y \tilde{\Pi}_{cs} T_c} \quad (\mathbf{u}, \underbrace{T \mathbf{e}_y}) = (v, T)$$

$$u_s \cdot \Omega_s E_y \tilde{\Pi}_{cs} T_c = u_s^y \Omega_s \tilde{\Pi}_{cs} T_c$$

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$$y_c \cdot M U_s \Pi_{cs} T_c = -y_c \cdot G^T \Omega_s U_s \Pi_{cs} T_c$$

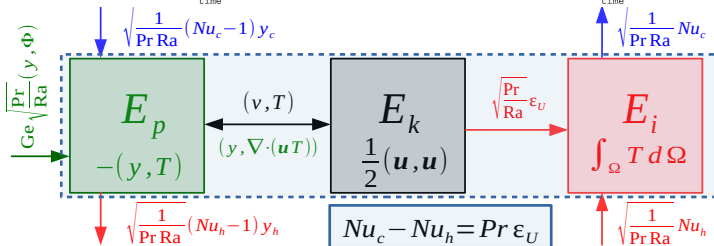
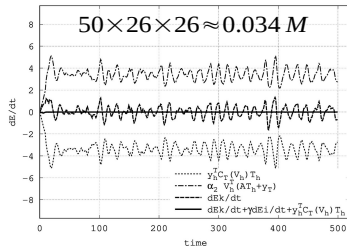
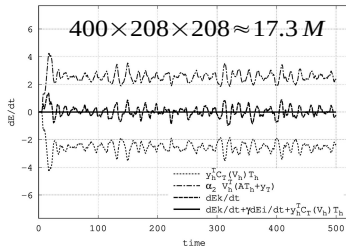
$$= -u_s^y \Omega_s \Pi_{cs} T_c$$



$$\tilde{\Pi}_{cs} = \Pi_{cs}$$

Buoyancy-driven flows

Discrete formulation on **staggered** grids



Buoyancy-driven flows

Discrete formulation on **collocated** grids

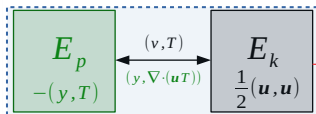
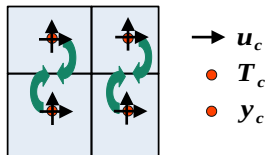
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + T \mathbf{e}_y \quad \text{Boussinesq} \quad \nabla \cdot \mathbf{u} = 0$$

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$$(\mathbf{v}, T) = \int_{\Omega} \mathbf{v} T d\Omega$$

$$T \mathbf{e}_y \quad (\mathbf{u}, T \mathbf{e}_y) = (\mathbf{v}, T)$$

$$(\mathbf{u} \cdot \nabla) T \quad (y, (\mathbf{u} \cdot \nabla) T) = -(\mathbf{v}, T)$$



Buoyancy-driven flows

Discrete formulation on **collocated** grids

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \sqrt{\frac{\text{Pr}}{\text{Ra}}} \nabla^2 \mathbf{u} - \nabla p + T \mathbf{e}_y \quad \text{Boussinesq} \quad \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \sqrt{\frac{1}{\text{Pr Ra}}} \nabla^2 T + \text{Ge} \sqrt{\frac{\text{Pr}}{\text{Ra}}} \Phi \quad \Phi = \|\nabla \mathbf{u}\|^2 \varepsilon_U = \int_{\Omega} \Phi d\Omega$$

$$(\mathbf{v}, T) = \int_{\Omega} \mathbf{v} T d\Omega$$

$$T \mathbf{e}_y$$

$$(\mathbf{u}, T \mathbf{e}_y) = (\mathbf{v}, T)$$

$$(\mathbf{u} \cdot \nabla) T$$

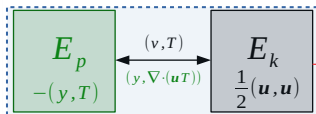
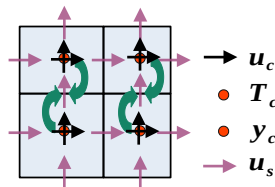
$$(y, (\mathbf{u} \cdot \nabla) T) = -(\mathbf{v}, T)$$

$$M U_s \Pi_{cs} T_c$$

$$y_c \cdot M U_s \Pi_{cs} T_c = -y_c \cdot G^T \Omega_s U_s \Pi_{cs} T_c$$

$$= -\mathbf{u}_s^y \Omega_s \Pi_{cs} T_c$$

$$\mathbf{u}_s = \Gamma_{cs} \mathbf{u}_c$$



Buoyancy-driven flows

Discrete formulation on **collocated** grids

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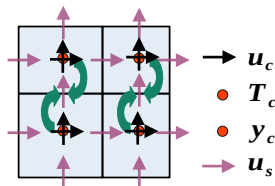
$$\underbrace{T \mathbf{e}_y}_{E_y \tilde{\Pi}_{cc} T_c} \quad (\mathbf{u}, \underbrace{T \mathbf{e}_y}) = (v, T)$$

$$u_c \cdot \Omega_c E_y \tilde{\Pi}_{cc} T_c = u_c^y \Omega_c \tilde{\Pi}_{cc} T_c$$

$$\underbrace{(\mathbf{u} \cdot \nabla) T}_{M U_s \Pi_{cs} T_c} \quad (y, \underbrace{(\mathbf{u} \cdot \nabla) T}) = -(v, T)$$

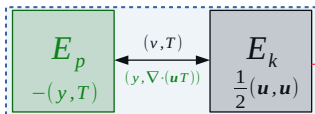
$$y_c \cdot M U_s \Pi_{cs} T_c = -y_c \cdot G^T \Omega_s U_s \Pi_{cs} T_c$$

$$= -u_s^y \Omega_s \Pi_{cs} T_c$$



$$u_s = \Gamma_{cs} u_c$$

$$\tilde{\Pi}_{cc} = I_c$$



Buoyancy-driven flows

Discrete formulation on **collocated** grids

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$$(v, T) = \int_{\Omega} v T d\Omega$$

$$T \mathbf{e}_y \quad (\mathbf{u}, T \mathbf{e}_y) = (v, T)$$

$$E_y \tilde{\Pi}_{cc} T_c \quad u_c \cdot \Omega_c E_y \tilde{\Pi}_{cc} T_c = u_c^y \Omega_c \tilde{\Pi}_{cc} T_c$$

$$(\mathbf{u} \cdot \nabla) T \quad (y, (\mathbf{u} \cdot \nabla) T) = -(v, T)$$

$$M U_s \Pi_{cs} T_c \quad y_c \cdot M U_s \Pi_{cs} T_c = -y_c \cdot G^T \Omega_s U_s \Pi_{cs} T_c$$

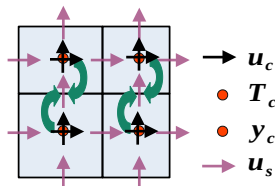
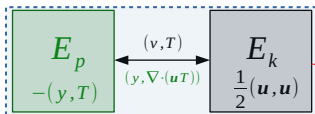
$$= -u_s^y \Omega_s \Pi_{cs} T_c$$

$$= -u_c^y \cdot \Gamma_{cs}^T \Omega_s \Pi_{cs} T_c$$

$$= -u_c^y \cdot \Omega_c \Gamma_{sc} \Pi_{cs} T_c$$

$$u_s = \Gamma_{cs} u_c$$

$$\tilde{\Pi}_{cc} = I_c$$



Buoyancy-driven flows

Discrete formulation on **collocated** grids

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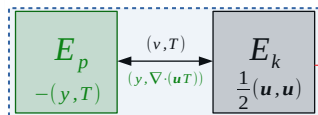
$$(v, T) = \int_{\Omega} v T d\Omega$$

$$\underbrace{T \mathbf{e}_y}_{E_y \tilde{\Pi}_{cc} T_c} \quad (\mathbf{u}, \underbrace{T \mathbf{e}_y}) = (v, T)$$

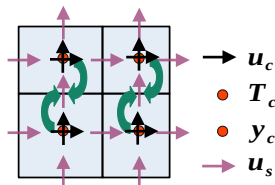
$$u_c \cdot \Omega_c E_y \tilde{\Pi}_{cc} T_c = u_c^y \Omega_c \tilde{\Pi}_{cc} T_c$$

$$\underbrace{(\mathbf{u} \cdot \nabla) T}_{M U_s \Pi_{cs} T_c} \quad (y, \underbrace{(\mathbf{u} \cdot \nabla) T}) = -(v, T)$$

$$y_c \cdot M U_s \Pi_{cs} T_c = -y_c \cdot G^T \Omega_s U_s \Pi_{cs} T_c$$



$$\begin{aligned} &= -u_s^y \Omega_s \Pi_{cs} T_c \\ &= -u_c^y \cdot \Gamma_{cs}^T \Omega_s \Pi_{cs} T_c \\ &= -u_c^y \cdot \Omega_c \Gamma_{sc} \Pi_{cs} T_c \end{aligned}$$



$$u_s = \Gamma_{cs} u_c$$

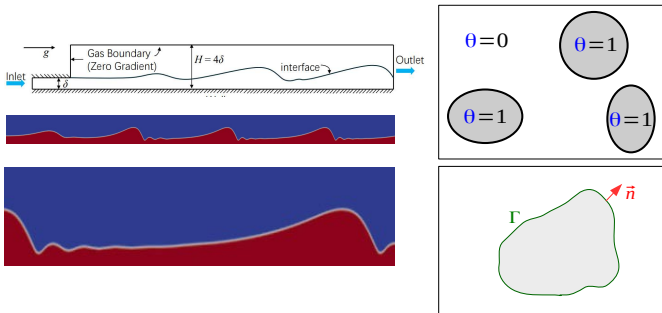
~~$$\tilde{\Pi}_{cc} = I_c$$~~

$$\tilde{\Pi}_{cc} = \Gamma_{sc} \Pi_{cs}$$

Multiphase flows: continuous formulation¹⁰

$$\text{Multiphase} \left\{ \begin{array}{l} \rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla) \vec{u} = \nabla \cdot (2\mu S(\vec{u})) - \nabla p + \gamma \kappa \cdot \vec{n} \quad \nabla \cdot \vec{u} = 0 \\ \frac{\partial \theta}{\partial t} + (\vec{u} \cdot \nabla) \theta = 0 \end{array} \right. \quad \leftarrow \text{essential idea of the \textbf{Convervative Level Set (CLS)} proposed by Olsson and Kreiss (JCP'05)}$$

$$\kappa = -\nabla \cdot \vec{n} \quad \text{where} \quad \vec{n} = \frac{\nabla \theta}{|\nabla \theta|}$$

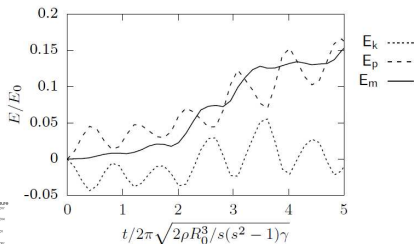
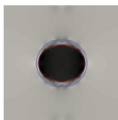
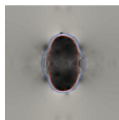
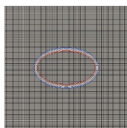


¹⁰N.Valle, F.X.Trias, and J.Castro. *An energy-preserving level set method for multiphase flows*. **Journal of Computational Physics**, 400(1):108991, 2020

Multiphase flows: continuous formulation¹⁰

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$$\kappa = -\nabla \cdot \vec{n} \quad \text{where} \quad \vec{n} = \frac{\nabla \theta}{|\nabla \theta|}$$



Mechanical energy (kinetic+potential) is not preserved!

¹⁰N.Valle, F.X.Trias, and J.Castro. *An energy-preserving level set method for multiphase flows*. **Journal of Computational Physics**, 400(1):108991, 2020

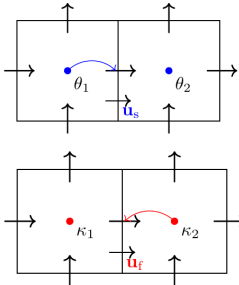
Multiphase flows: conservative discrete formulation¹⁰

Multiphase $\left\{ \begin{array}{l} \rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla) \vec{u} = \nabla \cdot (2\mu S(\vec{u})) - \nabla p + \gamma \kappa \cdot \vec{n} \\ \frac{\partial \theta}{\partial t} + (\vec{u} \cdot \nabla) \theta = 0 \end{array} \right.$ $\nabla \cdot \vec{u} = 0$

essential idea of the **Conservative Level Set (CLS)** proposed by Olsson and Kreiss (JCP'05)

$\kappa = -\nabla \cdot \vec{n}$ where $\vec{n} = \frac{\nabla \theta}{|\nabla \theta|}$

Surface tension



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An energy-preserving level set method for multiphase flows

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ABSTRACT

The computation of multiphase flows presents a subtle energetic equilibrium between potential (i.e., surface) and kinetic energies. The use of traditional interface-capturing schemes provides no control over such a dynamic balance. In the spirit of the well-known symmetry-preserving and mimetic schemes, whose physics-compatible discretizations rely upon preserving the underlying mathematical structures of the space, we identify the corresponding structure and propose a new discretization strategy for curvature. The new scheme ensures conservation of mechanical energy (i.e., surface plus kinetic) up to temporal integration. Inviscid numerical simulations are performed to show the robustness of such a method.

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Multiphase flows: conservative discrete formulation¹⁰

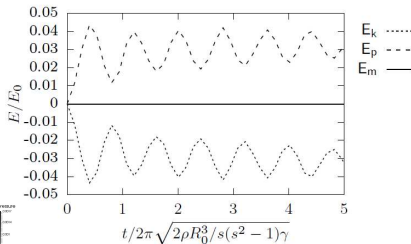
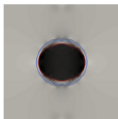
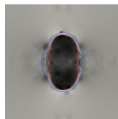
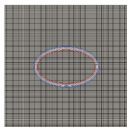
Multiphase

$$\left\{ \begin{array}{l} \rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla) \vec{u} = \nabla \cdot (2\mu S(\vec{u})) - \nabla p + \gamma \kappa \cdot \vec{n} \quad \nabla \cdot \vec{u} = 0 \\ \frac{\partial \theta}{\partial t} + (\vec{u} \cdot \nabla) \theta = 0 \end{array} \right.$$

essential idea of the **Conservative Level Set (CLS)** proposed by Olsson and Kreiss (JCP'05)

$\kappa = -\nabla \cdot \vec{n}$ where $\vec{n} = \frac{\nabla \theta}{|\nabla \theta|}$

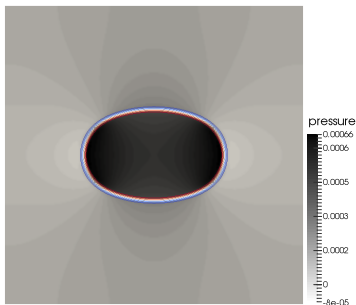
Surface tension



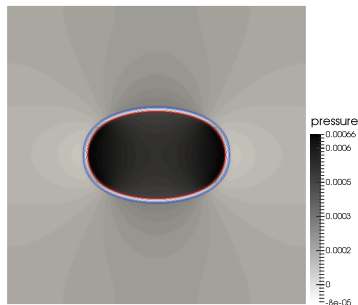
Mechanical energy (kinetic+potential) is preserved!

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Multiphase flows: conservative discrete formulation¹⁰



Non-symmetry-preserving!



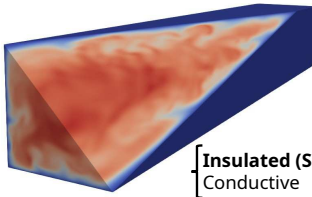
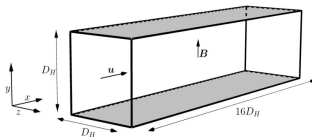
Symmetry-preserving!

¹⁰N.Valle, F.X.Trias, and J.Castro. *An energy-preserving level set method for multiphase flows*. **Journal of Computational Physics**, 400(1):108991, 2020

Magnetohydrodynamics¹¹

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = \frac{1}{\text{Re}} \nabla^2 \vec{u} - \nabla p + \frac{Ha^2}{\text{Re}} (\vec{J} \times \vec{B}) \quad \nabla \cdot \vec{u} = 0$$

$$\vec{J} = \vec{u} \times \vec{B} - \nabla \varphi \quad \nabla \cdot \vec{J} = 0$$



Insulated (Shercliff)
Conductive

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A checkerboard-free, symmetry-preserving, conservative method for magnetohydrodynamics

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ABSTRACT

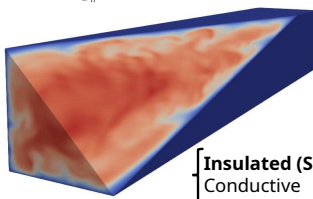
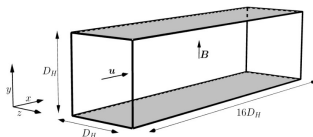
Many industrial applications, such as nuclear fusion reactor design and aeronautics, require accurate numerical simulation tools, that conserve physical properties, for magnetohydrodynamic (MHD) flows at low magnetic Reynolds numbers. The interaction between the magnetic field and the conductive fluid often results in complex flow phenomena, which require conservative discretization schemes. These schemes minimize numerical dispersion to improve accuracy, but can lead to the checkerboard problem if applied on collocated grid arrangements, which are necessary in complex geometries. An algorithm is constructed to simulate MHD flows, which relies on the symmetry-preserving discretization to allow minimal numerical dispersion. Moreover, it quantifies the checkerboard problem, its dynamical balance with numerical dissipation, only when necessary. Results for a laminar MHD Taylor-Green vortex case show the importance of the symmetry-preserving discretization, for which the face-to-cell interpolation method is highlighted. Moreover, it is shown that the solver is virtually free from numerical dissipation on uniform meshes, and effectively counteracts checkerboarding on skew meshes, where other solvers encounter difficulties. Turbulent MHD duct flow cases show the ability of the solver to generate

¹¹ J.A.Hopman, J.Rigola, F.X.Trias. *A checkerboard-free, symmetry-preserving, conservative method for magneto-hydrodynamics*, **Journal of Computational Physics**, 545:114475, 2026.

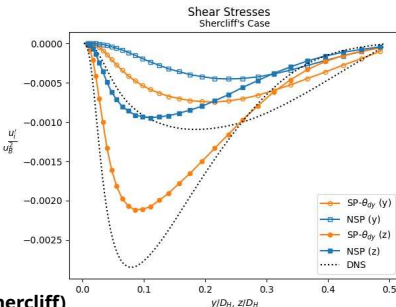
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$$\vec{J} = \vec{u} \times \vec{B} - \nabla \phi \quad \nabla \cdot \vec{J} = 0$$



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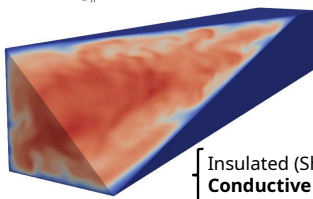
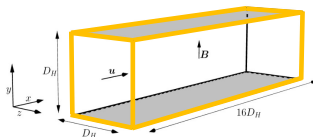


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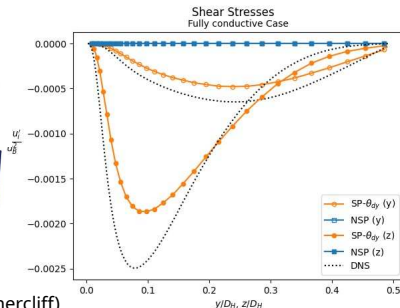
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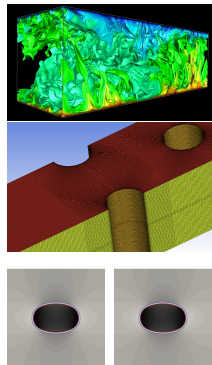
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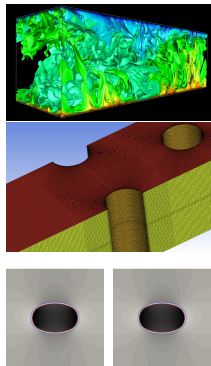
Take-away messages

- **Preserving symmetries** is the key point for **reliable LES/DNS** simulations for both academic and industrial problems.



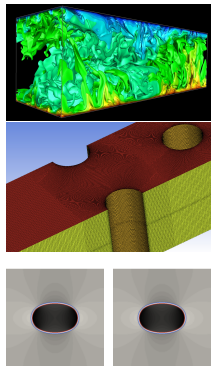
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- **Preserving symmetries** is the key point for **reliable LES/DNS** simulations for both academic and industrial problems.
- We usually focus on quadratic invariants (kinetic energy or enstrophy) but **energy exchanges** are also crucial for multiphysics.



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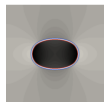
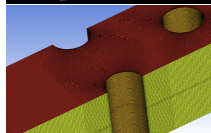
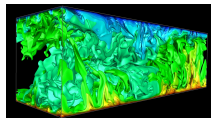
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- We usually focus on quadratic invariants (kinetic energy or enstrophy) but **energy exchanges** are also crucial for multiphysics.
- Challenging numbers on very **coarse** and **poor quality** meshes.
- **Open-source** platforms helps to share and disseminate ideas.

<https://github.com/janneshopman/RKSymFoam>



Open  FOAM

