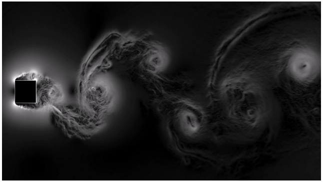
Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

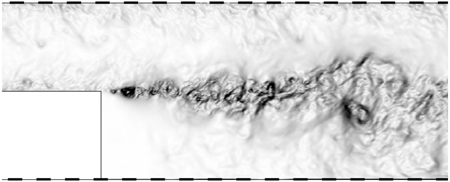
Paving the way for DNS and LES on unstructured grids

F.Xavier Trias[1], Andrey Gorobets[2], Assensi Oliva[1]

[1]Heat and Mass Transfer Technological Center, Technical University of Catalonia

[2]Keldysh Institute of Applied Mathematics of RAS, Russia





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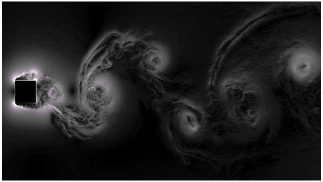

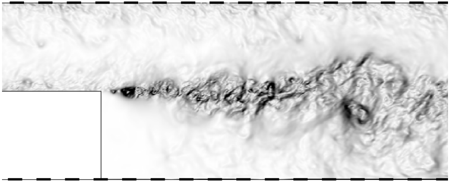

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Paving the (right?) way for DNS and LES on unstructured grids: fully conservative collocated/staggered discretizations

F.Xavier Trias[1], Andrey Gorobets[2], Assensi Oliva[1]

[1]Heat and Mass Transfer Technological Center, Technical University of Catalonia

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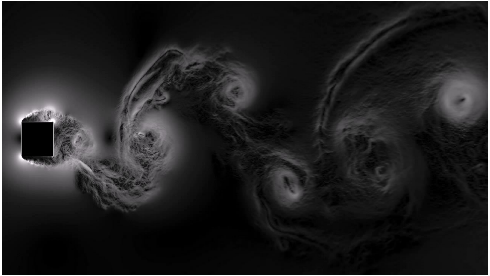
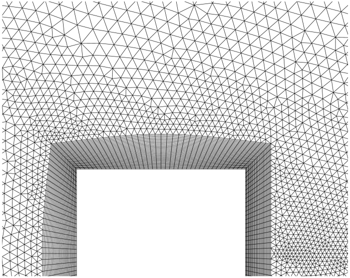
Contents

Motivation

Research question #1:

- Can we construct numerical discretizations of the Navier-Stokes equations suitable for **complex geometries**, such that the **symmetry properties** are exactly preserved?





DNS[1] of the turbulent flow around a square cylinder at $Re = 22000$

[1] F.X.Trias, A.Gorobets, A.Oliva. *Turbulent flow around a square cylinder at Reynolds number 22000: a DNS study*, **Computers&Fluids**, 123:87-98, 2015.

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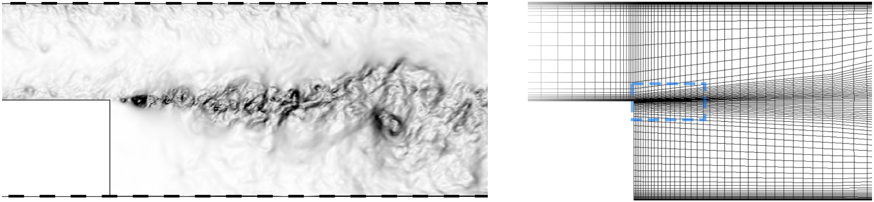


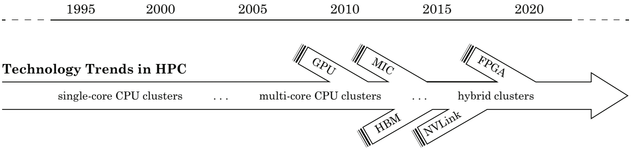
DNS[2] of backward-facing step at $Re_\tau = 395$ and expansion ratio 2

[2] A.Pont-Vílchez, F.X.Trias, A.Gorobets, A.Oliva. *DNS of Backward-Facing Step flow at* $Re_\tau = 395$ *and expansion ratio 2*. **Journal of Fluid Mechanics**, 863:341-363, 2019.

Motivation

Research question #2:

- How can we develop **portable** and **efficient** CFD codes for large-scale simulations on modern supercomputers?



[3] X.Álvarez, A.Gorobets, F.X.Trias, R.Borrell, G.Oyarzun. *HPC^2 - a fully portable algebra-dominant framework for heterogeneous computing. Application to CFD*. **Computers & Fluids**, 173:285-292, 2018.

[4] X.Álvarez, A.Gorobets, F.X.Trias. *A hierarchical parallel implementation for heterogeneous computing. Application to algebra-based CFD simulations on hybrid supercomputers*. **Computers & Fluids**, 214:104768, 2021.

[5] X.Álvarez, A.Gorobets, F.X.Trias, A.Oliva. *NUMA-aware strategies for the efficient execution of CFD simulations on CPU supercomputers* **ParCFD2021**. Don't miss it!

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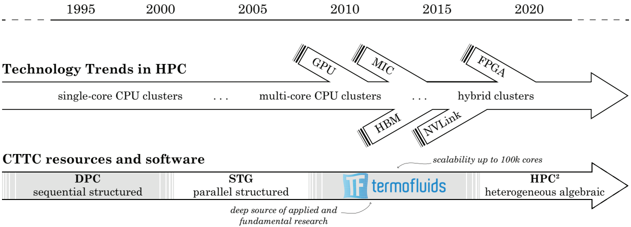

HPC²: portable, algebra-based framework[3] for heterogeneous computing is being developed[4]. Traditional stencil-based data and sweeps are replaced by algebraic structures (sparse matrices and vectors) and kernels. NUMA-aware execution strategies for CFD are presented in this conference[5].

[3] X.Álvarez, A.Gorobets, F.X.Trias, R.Borrell, G.Oyarzun. *HPC² - a fully portable algebra-dominant framework for heterogeneous computing. Application to CFD*. **Computers & Fluids**, 173:285-292, 2018.

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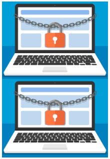
Motivation

Frequently used general purpose CFD codes:

- STAR-CCM+
- ANSYS-FLUENT
- Code-Saturne
- OpenFOAM



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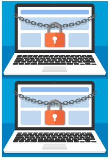
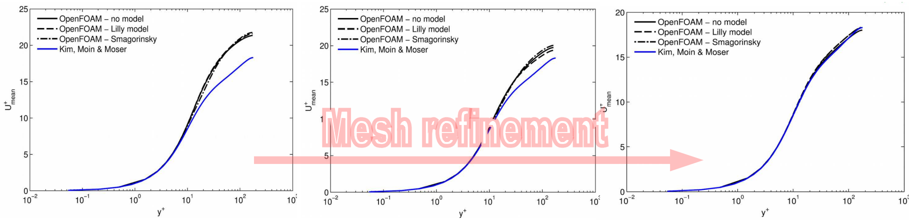


Main common characteristics of LES in such codes:

- **Unstructured finite volume** method, **collocated** grid
- Second-order spatial and temporal discretisation
- Eddy-viscosity type LES models

Motivation

Open∇FOAM® LES[6] results of a turbulent channel for at $Re_\tau = 180$



$13 \times 76 \times 20$ — $\Delta x^+ = 90, \Delta y^+_{wall} = 0.5, \Delta z^+ = 30$

$19 \times 78 \times 28$ — $\Delta x^+ = 60, \Delta y^+_{wall} = 0.5, \Delta z^+ = 20$

$38 \times 78 \times 57$ — $\Delta x^+ = 30, \Delta y^+_{wall} = 0.5, \Delta z^+ = 10$

---

[6] E.M.J.Komen, L.H.Camilo, A.Shams, B.J.Geurts, B.Koren. *A quantification method for numerical dissipation in quasi-DNS and under-resolved DNS, and effects of numerical dissipation in quasi-DNS and under-resolved DNS of turbulent channel flows*, **Journal of Computational Physics**, 345, 565-595, 2017.

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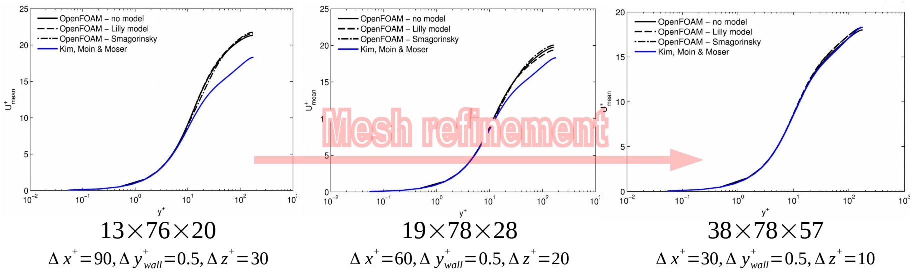

- Are LES results are merit of the SGS model?

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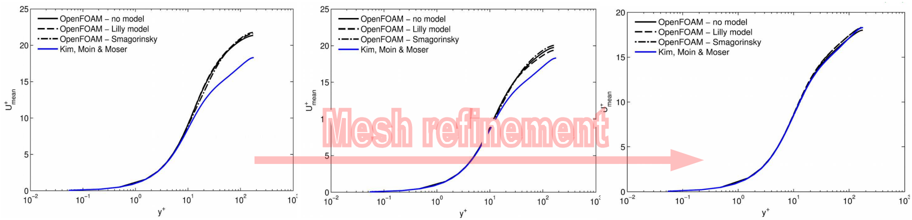

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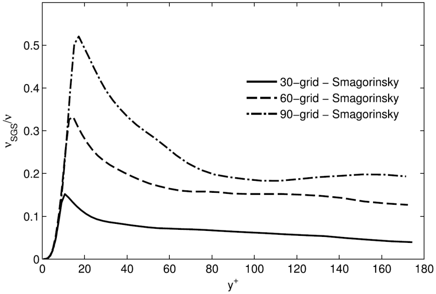
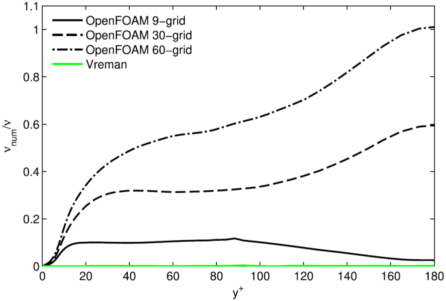
- Are LES results are merit of the SGS model? Apparently **NOT**!!! ✗

[6] E.M.J.Komen, L.H.Camilo, A.Shams, B.J.Geurts, B.Koren. *A quantification method for numerical dissipation in quasi-DNS and under-resolved DNS, and effects of numerical dissipation in quasi-DNS and under-resolved DNS of turbulent channel flows*, **Journal of Computational Physics**, 345, 565-595, 2017.

Motivation

OpenFOAM® LES[7] results of a turbulent channel for at $Re_\tau = 180$





$$\nu_{num} \neq 0$$

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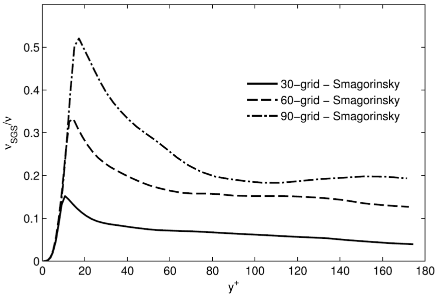

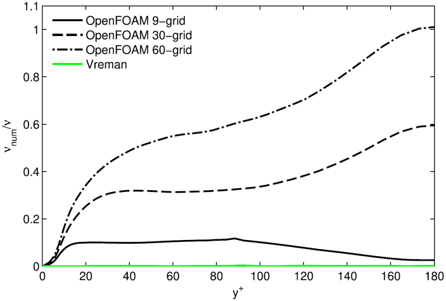

$$\nu_{SGS} < \nu_{num} \neq 0$$

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Symmetry-preserving discretization

Continuous

$$\frac{\partial \boldsymbol{u}}{\partial t} + C(\boldsymbol{u}, \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u} - \nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Symmetry-preserving discretization

Continuous	Discrete
$\frac{\partial \boldsymbol{u}}{\partial t} + C(\boldsymbol{u}, \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u} - \nabla p$	$\Omega \frac{d\boldsymbol{u}_h}{dt} + \mathsf{C}(\boldsymbol{u}_h)\, \boldsymbol{u}_h = \mathsf{D}\boldsymbol{u}_h - \mathsf{G}\boldsymbol{p}_h$
$\nabla \cdot \boldsymbol{u} = 0$	$\mathsf{M}\boldsymbol{u}_h = \boldsymbol{0}_h$

Symmetry-preserving discretization

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$$\frac{\partial \boldsymbol{u}}{\partial t} + C(\boldsymbol{u}, \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u} - \nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \int_\Omega \boldsymbol{a}\boldsymbol{b} d\Omega$$

Discrete

$$\Omega \frac{d \boldsymbol{u}_h}{dt} + \mathsf{C}(\boldsymbol{u}_h)\, \boldsymbol{u}_h = \mathsf{D}\boldsymbol{u}_h - \mathsf{G}\boldsymbol{p}_h$$

$$\mathsf{M}\boldsymbol{u}_h = \boldsymbol{0}_h$$

$$\langle \boldsymbol{a}_h, \boldsymbol{b}_h \rangle_h = \boldsymbol{a}_h^T \Omega \boldsymbol{b}_h$$

Symmetry-preserving discretization

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$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \int_\Omega \boldsymbol{a}\boldsymbol{b}\, d\Omega$	$\langle \boldsymbol{a}_h, \boldsymbol{b}_h \rangle_h = \boldsymbol{a}_h^T \Omega \boldsymbol{b}_h$
$\langle C(\boldsymbol{u}, \varphi_1), \varphi_2 \rangle = - \langle C(\boldsymbol{u}, \varphi_2), \varphi_1 \rangle$	$\mathsf{C}(\boldsymbol{u}_h) = -\mathsf{C}^T(\boldsymbol{u}_h)$

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$\langle C(\boldsymbol{u}, \varphi_1), \varphi_2 \rangle = -\langle C(\boldsymbol{u}, \varphi_2), \varphi_1 \rangle$	$\mathsf{C}(\boldsymbol{u}_h) = -\mathsf{C}^T(\boldsymbol{u}_h)$
$\langle \nabla \cdot \boldsymbol{a}, \varphi \rangle = -\langle \boldsymbol{a}, \nabla \varphi \rangle$	$\Omega \mathsf{G} = -\mathsf{M}^T$

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$\frac{\partial \boldsymbol{u}}{\partial t} + C(\boldsymbol{u}, \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u} - \nabla p$	$\Omega \frac{d\boldsymbol{u}_h}{dt} + \mathsf{C}(\boldsymbol{u}_h)\,\boldsymbol{u}_h = \mathsf{D}\boldsymbol{u}_h - \mathsf{G}\boldsymbol{p}_h$
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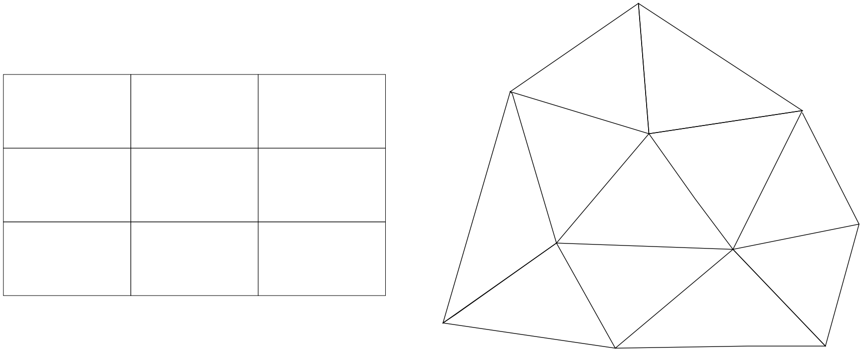
$\langle \nabla \cdot \boldsymbol{a}, \varphi \rangle = - \langle \boldsymbol{a}, \nabla \varphi \rangle$	$\Omega \mathsf{G} = -\mathsf{M}^T$
$\left\langle \nabla^2 \boldsymbol{a}, \boldsymbol{b} \right\rangle = \left\langle \boldsymbol{a}, \nabla^2 \boldsymbol{b} \right\rangle$	$\mathsf{D} = \mathsf{D}^T \quad def -$

Collocated vs staggered



Collocated vs staggered

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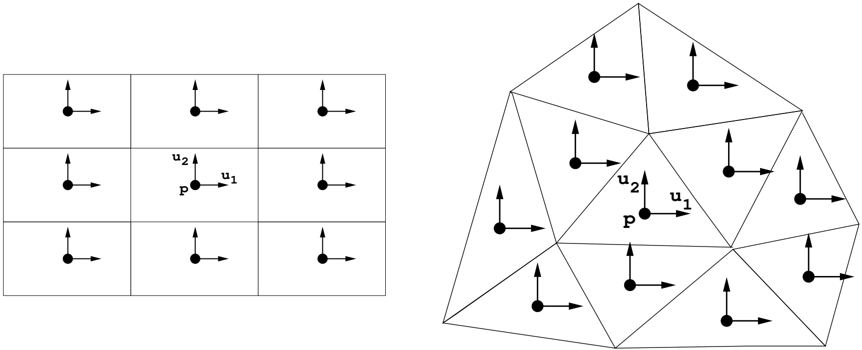

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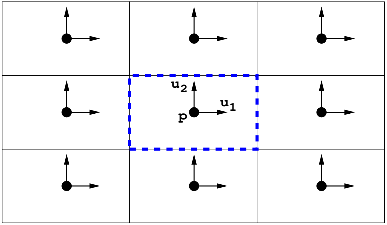

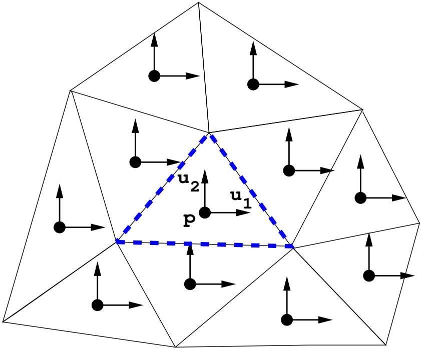

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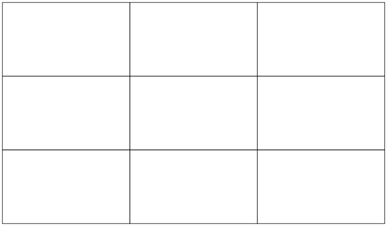

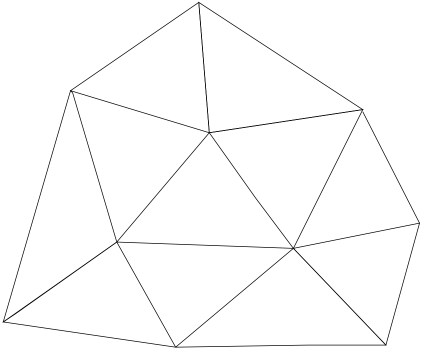

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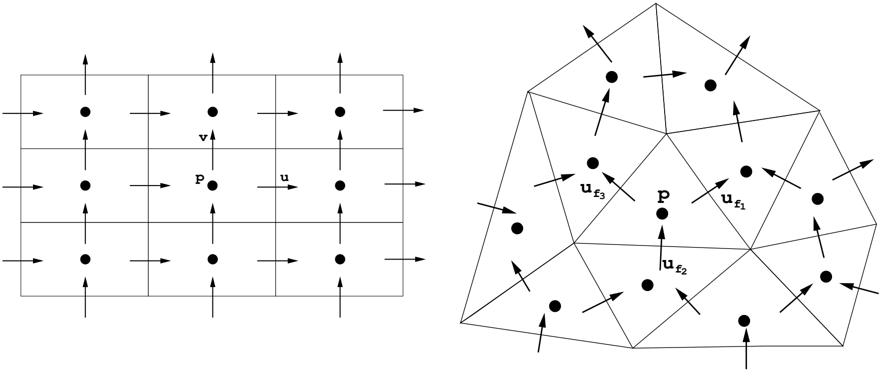

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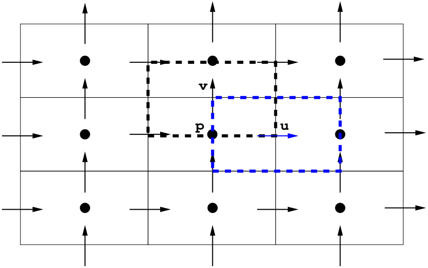
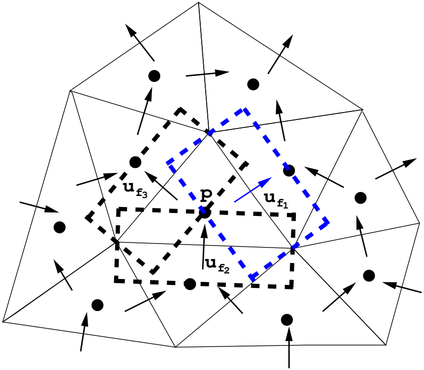
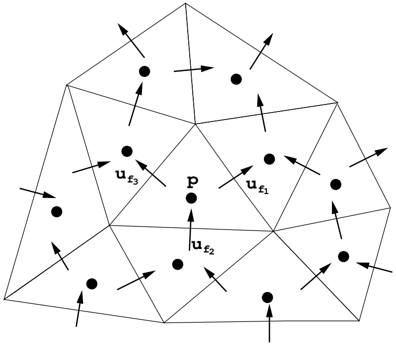




Why staggered?

$$\boxed{\Omega_s \frac{d\boldsymbol{u}_s}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_s = \mathsf{D}\boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c; \quad \mathsf{M}\boldsymbol{u}_s = \mathbf{0}_c}$$



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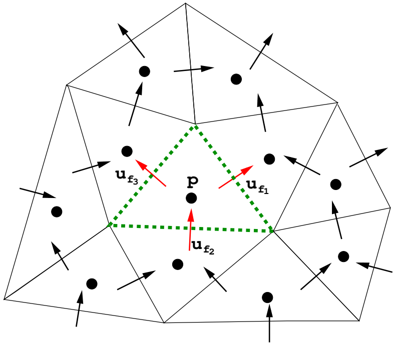
Let's consider we have $\boldsymbol{u}_s$ such as

$$\mathsf{M}\boldsymbol{u}_s \neq \mathbf{0}_c$$



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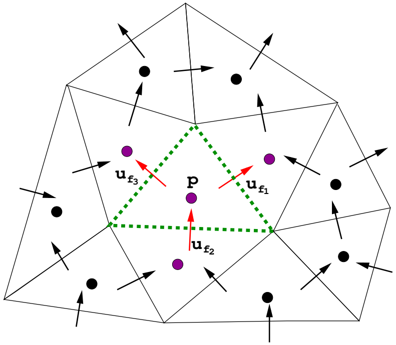
then, we can easily project $\boldsymbol{u}_s$

$$\boldsymbol{u}_s = \quad \boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c$$



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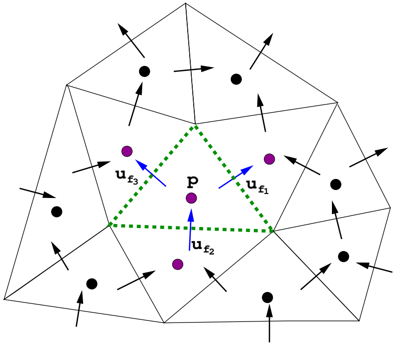

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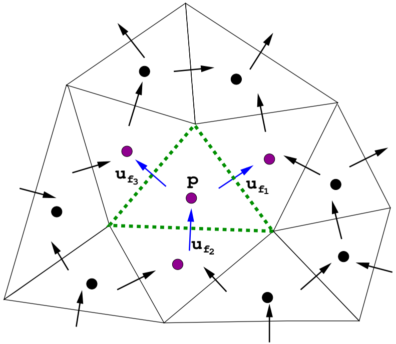
Finally, this leads to a Poisson eq.

$$\boxed{\mathsf{M}\mathsf{G}\boldsymbol{p}_c = \mathsf{M}\boldsymbol{u}_s}$$



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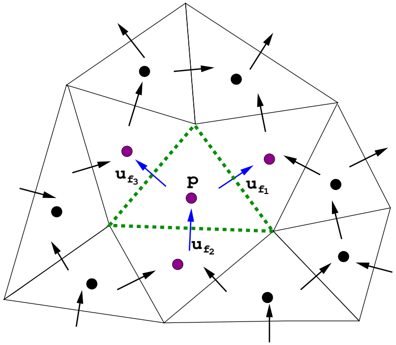


If $\Omega_s \mathsf{G} = -\mathsf{M}^T$

$$\langle \nabla \cdot \boldsymbol{a}, \varphi \rangle = -\langle \boldsymbol{a}, \nabla \varphi \rangle$$

Why staggered? Everything seems to be in the right place!

$$\boxed{\Omega_s \frac{d\boldsymbol{u}_s}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_s = \mathsf{D}\boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c; \quad \mathsf{M}\boldsymbol{u}_s = \mathbf{0}_c}$$

Let's consider we have $\boldsymbol{u}_s$ such as

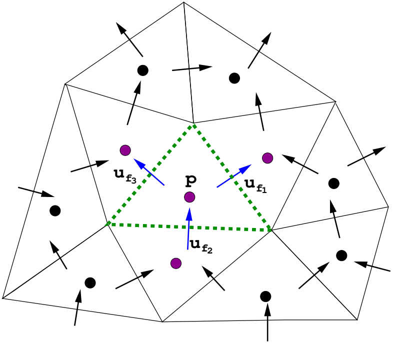
$$\mathsf{M}\boldsymbol{u}_s \neq \mathbf{0}_c$$

then, we can easily project $\boldsymbol{u}_s$

$$\mathsf{M}\boldsymbol{u}_s = \mathsf{M}(\boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c) = \mathbf{0}_c$$

Finally, this leads to a Poisson eq.

$$\boxed{\mathsf{M}\mathsf{G}\boldsymbol{p}_c = \mathsf{M}\boldsymbol{u}_s}$$

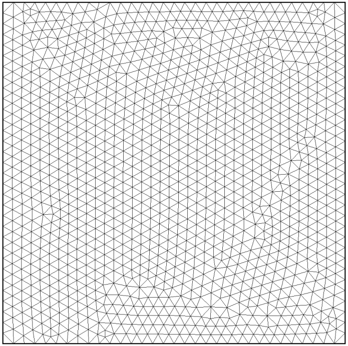


$$\text{If } \Omega_s\mathsf{G} = -\mathsf{M}^T \implies \langle \boldsymbol{u}_s, \mathsf{G}\boldsymbol{p}_c \rangle_h = \boldsymbol{u}_s^T \Omega_s \mathsf{G}\boldsymbol{p}_c = -(\mathsf{M}\boldsymbol{u}_s)^T \boldsymbol{p}_c = 0$$

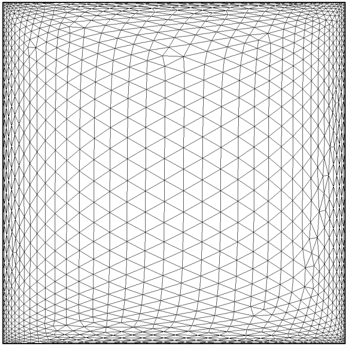
$$\langle \nabla \cdot \boldsymbol{a}, \varphi \rangle = -\langle \boldsymbol{a}, \nabla \varphi \rangle \implies \langle \boldsymbol{u}, \nabla p \rangle = -\langle \nabla \cdot \boldsymbol{u}, p \rangle = 0$$

But is this discrete Laplacian accurate?

Without stretching



With stretching



$$\nabla^2\varphi = f(x,y) \quad \text{with } f(x,y) = \nabla^2\left(k^{-2}\sin(kx)\sin(ky)\right) \text{ and } k = 25\pi$$

But is this discrete Laplacian accurate?

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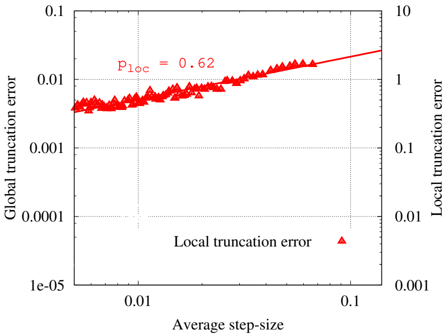

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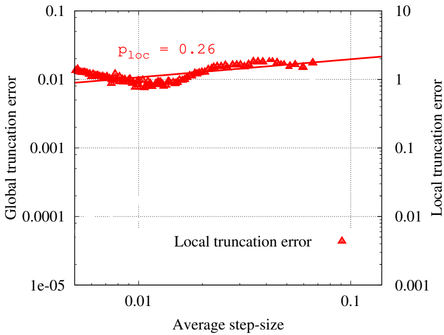
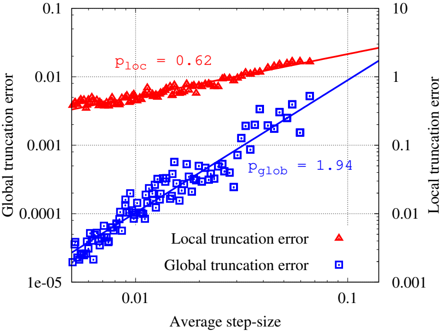


$$\nabla^2 \varphi = f(x,y) \quad \text{with } f(x,y) = \nabla^2(k^{-2}\sin(kx)\sin(ky)) \text{ and } k = 25\pi$$

But is this discrete Laplacian accurate?

Yes, even for distorted unstructured meshes! And symmetries are preserved!

Without stretching



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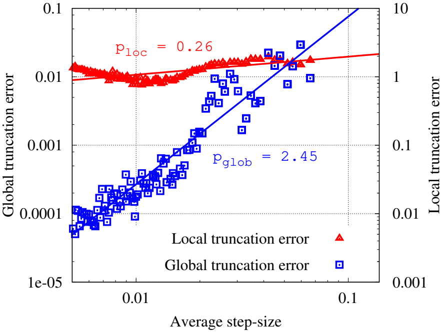


$$\nabla^2\varphi = f(x,y) \quad \text{with } f(x,y) = \nabla^2(k^{-2}\sin(kx)\sin(ky)) \text{ and } k = 25\pi$$

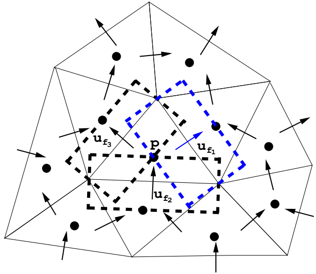
Then, why collocated arrangements are so popular?

- STAR-CCM+
- ANSYS-FLUENT
- Code-Saturne
- OpenFOAM

$$\boxed{\Omega_s \frac{d\boldsymbol{u}_s}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_s = \mathsf{D}\boldsymbol{u}_s - \mathsf{G}\boldsymbol{p}_c; \quad \mathsf{M}\boldsymbol{u}_s = \boldsymbol{0}_c}$$

In staggered meshes

- p-$\boldsymbol{u}_s$ coupling is naturally solved ✓
- $\mathsf{C}\left(\boldsymbol{u}_s\right)$ and D difficult to discretize ✗



Then, why collocated arrangements are so popular?

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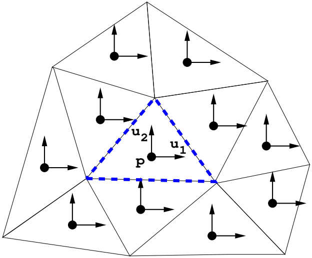
$$\boxed{\Omega_c \frac{d\boldsymbol{u}_c}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_c = \mathsf{D}\boldsymbol{u}_c - \mathsf{G}_c\boldsymbol{p}_c; \quad \mathsf{M}_c\boldsymbol{u}_c = \boldsymbol{0}_c}$$

In <u>collocated meshes</u>

- p-$\boldsymbol{u}_c$ coupling is cumbersome ✗
- $\mathsf{C}\left(\boldsymbol{u}_s\right)$ and D easy to discretize ✓
- Cheaper, less memory,... ✓



Then, why collocated arrangements are so popular?

Everything is easy except the pressure-velocity coupling...

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- Code-Saturne
- OpenFOAM

$$\boxed{\Omega_c \frac{d\boldsymbol{u}_c}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right)\boldsymbol{u}_c = \mathsf{D}\boldsymbol{u}_c - \mathsf{G}_c \boldsymbol{p}_c; \quad \mathsf{M}_c \boldsymbol{u}_c = \boldsymbol{0}_c}$$

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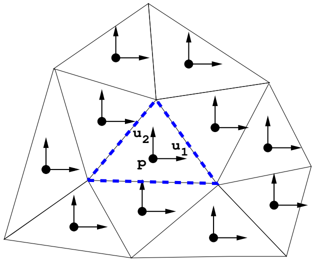
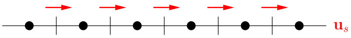


Pressure-velocity coupling on staggered grids

Works perfectly!



$\boldsymbol{u_s}$

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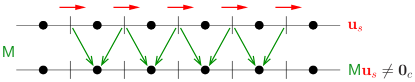

$\mathsf{M}\boldsymbol{u_s}$

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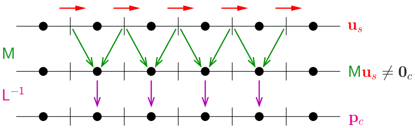

$$\underbrace{\mathsf{L}^{-1}\mathsf{M}\boldsymbol{u}_s}_{\boldsymbol{p}_c}$$

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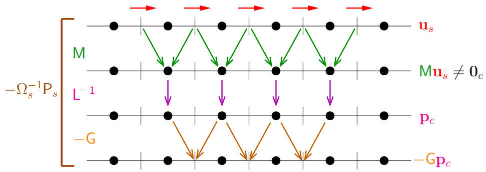

$$-\mathsf{G}\,\underbrace{\mathsf{L}^{-1}\mathsf{M}\boldsymbol{u_s}}_{\boldsymbol{p_c}}$$

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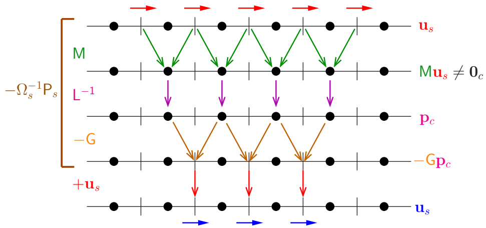

$$\boldsymbol{u_s} = \boldsymbol{u_s} - \mathsf{G}\underbrace{\mathsf{L}^{-1}\mathsf{M}\boldsymbol{u_s}}_{\boldsymbol{p_c}}$$

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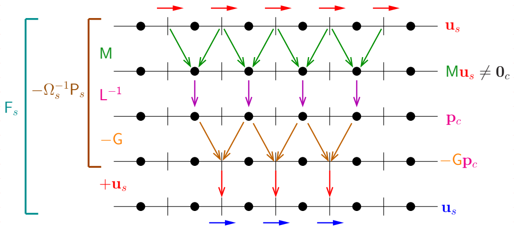

$$\boldsymbol{u}_s = \boldsymbol{u}_s - \mathsf{G}\underbrace{\mathsf{L}^{-1}\mathsf{M}\boldsymbol{u}_s}_{\boldsymbol{p}_c} = (I - \Omega_s^{-1}\mathsf{P}_s)\boldsymbol{u}_s = \mathsf{F}_s\boldsymbol{u}_s$$

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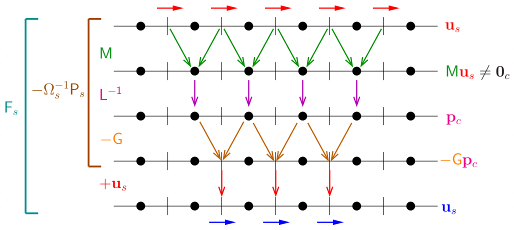

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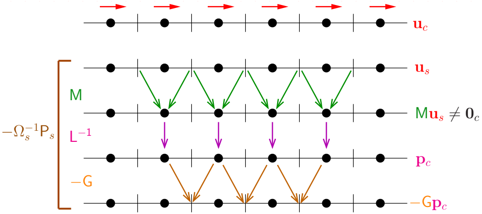
Pressure-velocity coupling on collocated grids

A vicious circle that cannot be broken...



$\boldsymbol{u}_c$

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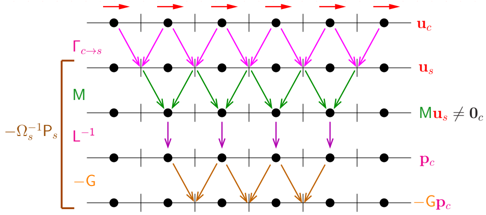

$\Gamma_{c\to s}\boldsymbol{u}_c$

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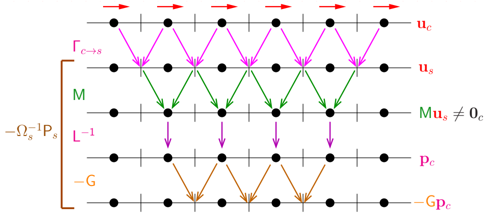

$$\mathsf{G}\mathsf{L}^{-1}\mathsf{M}\Gamma_{c\to s}\boldsymbol{u}_c$$

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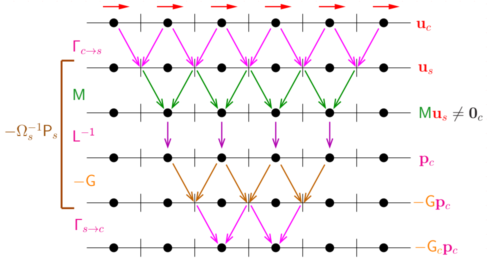

$$-\Gamma_{s\to c}\mathsf{G}\mathsf{L}^{-1}\mathsf{M}\Gamma_{c\to s}\boldsymbol{u}_c$$

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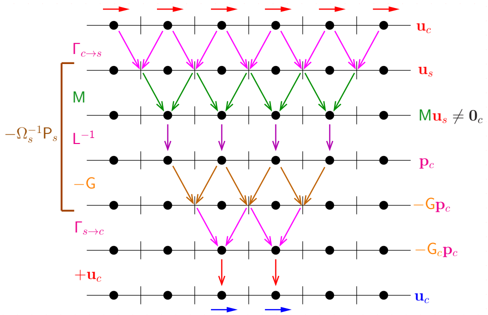

$$\boldsymbol{u}_c = \boldsymbol{u}_c - \Gamma_{s\to c}\mathsf{G}\mathsf{L}^{-1}\mathsf{M}\Gamma_{c\to s}\boldsymbol{u}_c$$

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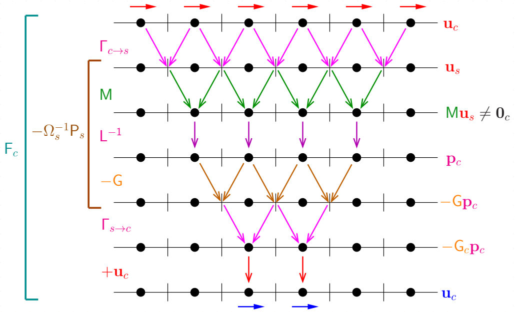

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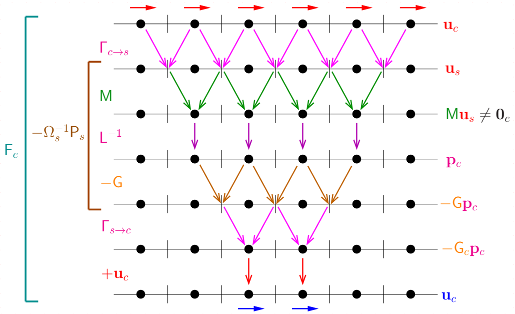

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To preserve symmetry we impose $\Gamma_{s\to c} = \Omega_c^{-1}\Gamma_{c\to s}^T\Omega_s$. This leads to

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where $\mathsf{L}_c = -\mathsf{M}\Gamma_{c\to s}\Omega_c^{-1}\Gamma_{c\to s}^T\mathsf{M}$ (wide-stencil discrete Laplacian).

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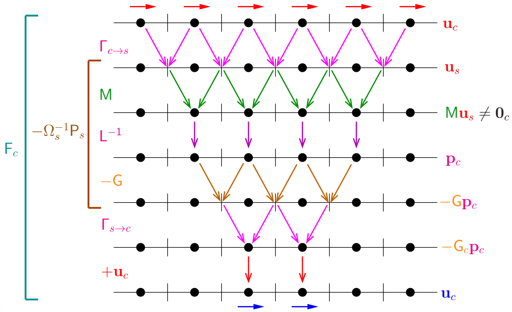

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Moreover, **contribution to kinetic enegy**: $\boldsymbol{p}_c(\mathsf{L} - \mathsf{L}_c)\boldsymbol{p}_c \neq 0$ ✗

Pressure-velocity coupling on collocated grids

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In summary[8]:

- Mass: $M\Gamma_{c\rightarrow s}\boldsymbol{u}_c = M\Gamma_{c\rightarrow s}\boldsymbol{u}_c - \mathsf{L}_c\mathsf{L}^{-1} M\Gamma_{c\rightarrow s}\boldsymbol{u}_c \approx \mathbf{0}_c$ ✗
- Energy: $\boldsymbol{p}_c\,(\mathsf{L} - \mathsf{L}_c)\,\boldsymbol{p}_c \neq 0$ ✗

[8] F.X.Trias, O.Lehmkuhl, A.Oliva, C.D.Pérez-Segarra, R.W.C.P.Verstappen. *Symmetry-preserving discretization of Navier-Stokes equations on collocated unstructured grids*, **Journal of Computational Physics**, 258 (1): 246-267, 2014.

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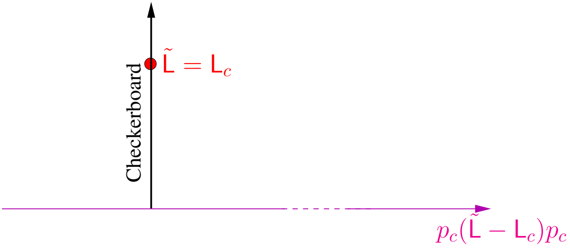

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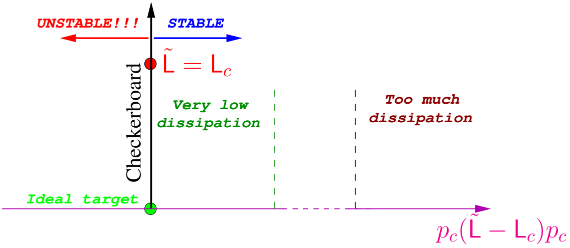

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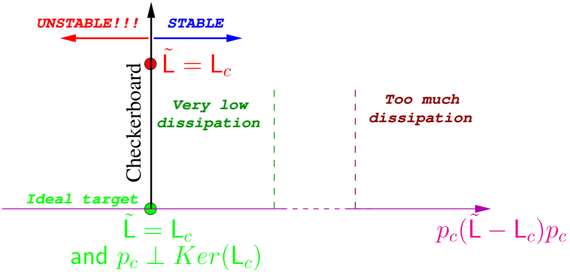

[8]Shashank, J.Larsson, G.Iaccarino. *A co-located incompressible Navier-Stokes solver with exact mass, momentum and kinetic energy conservation in the inviscid limit*, **Journal of Computational Physics**, 229: 4425-4430,2010.

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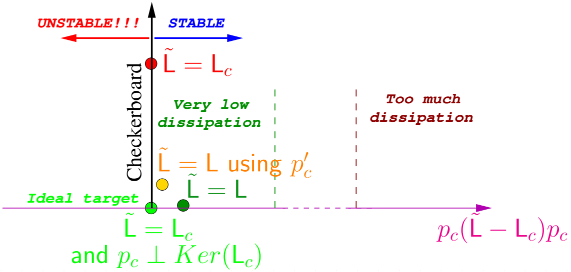

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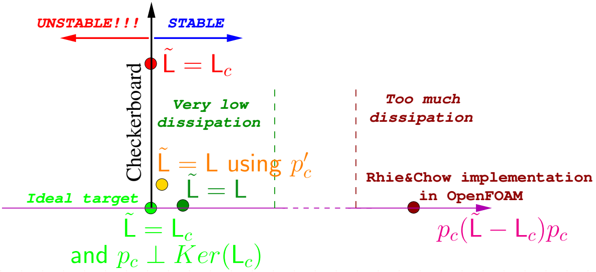

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Pressure-velocity coupling on collocated grids

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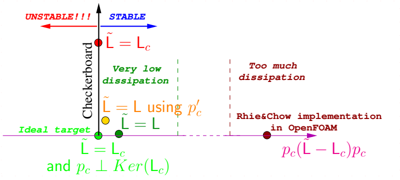

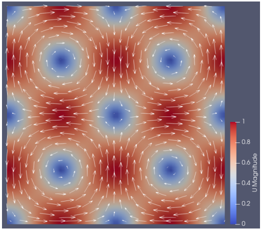

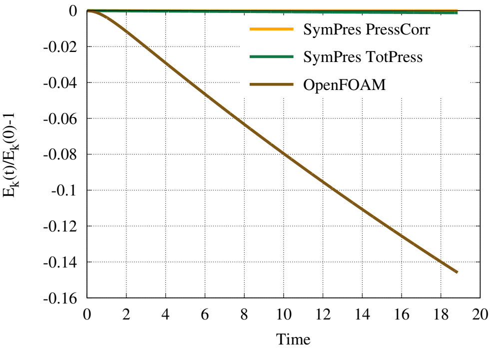

Results for an inviscid Taylor-Green vortex[9]

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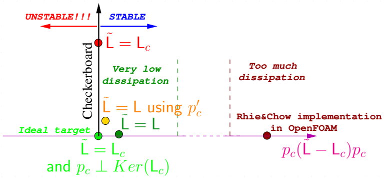
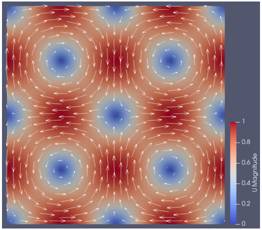
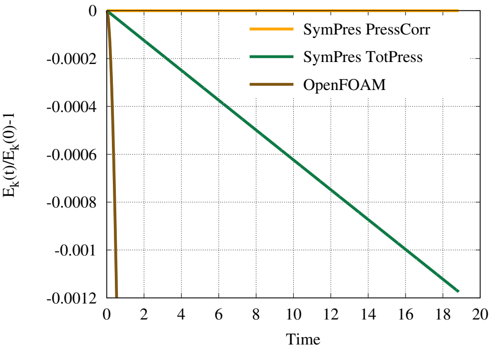






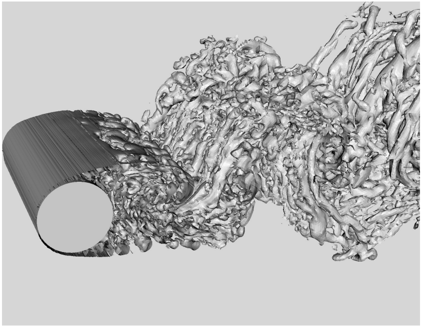
Results for an inviscid Taylor-Green vortex[9]

[9] E.Komen, J.A.Hopman, E.M.A.Frederix, F.X.Trias, R.W.C.P.Verstappen. "A symmetry-preserving second-order time-accurate PISO-based method". **Computers & Fluids**, (accepted).

Pressure-velocity coupling on collocated grids

Examples of simulations

Despite these inherent limitations, symmetry-preserving collocated formulation has been successfully used for DNS/LES simulations[10]:



[10] R.Borrell, O.Lehmkuhl, F.X.Trias, A.Oliva. *Parallel Direct Poisson solver for discretizations with one Fourier diagonalizable direction*. **Journal of Computational Physics**, 230:4723-4741, 2011.

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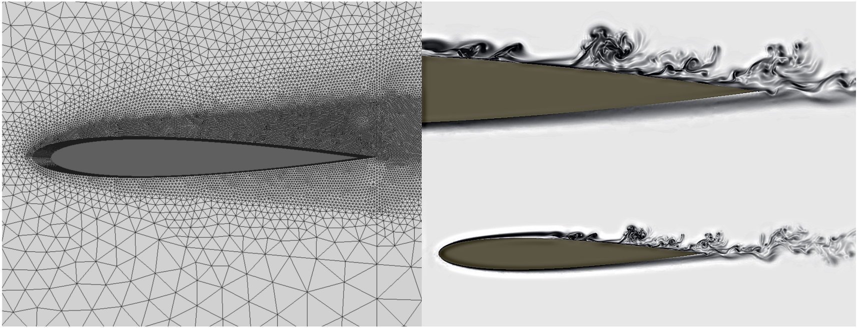
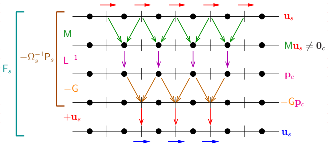
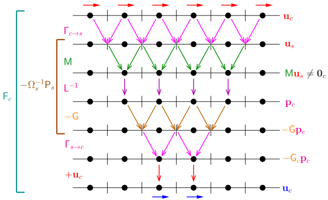


[10] F.X.Trias and O.Lehmkuhl. *A self-adaptive strategy for the time-integration of Navier-Stokes equations*. **Numerical Heat Transfer, part B**, 60(2):116-134, 2011.

Are staggered and collocated so different at the end?





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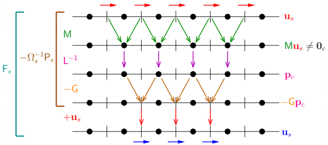

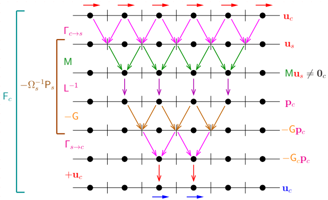

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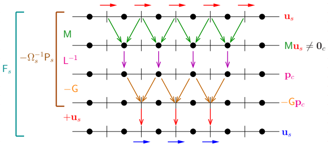

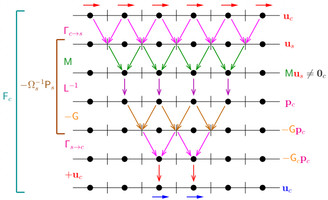

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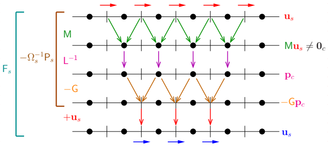

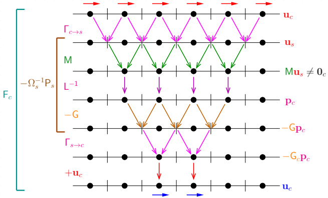

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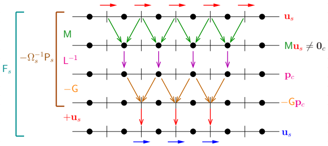

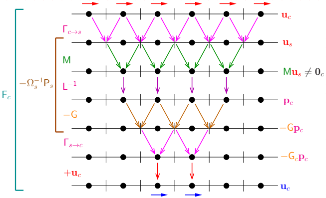

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Can we have a staggered formulation based only on collocated operators?

Then, it could be easily implemented in existing collocated codes such as OpenFOAM

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Similar approaches have been proposed in the literature before[11,12,13,14,15].

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Research question: then, why at the end collocated approach seems to be the winner?

- STAR-CCM+ — CD-adapco — SIEMENS
- ANSYS-FLUENT — ANSYS FLUENT
- Code-Saturne — CODE SATURNE — eDF — GPL
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Results for a turbulent channel flow at $Re_\tau = 180$

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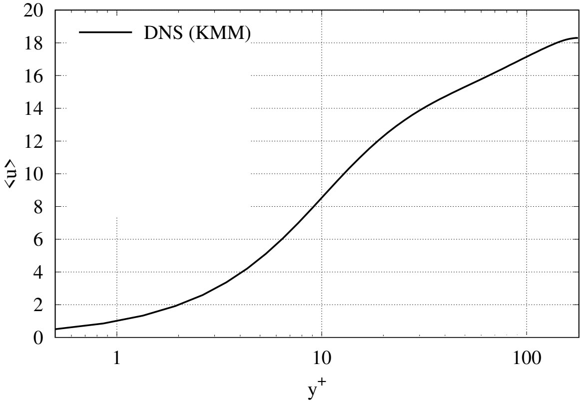

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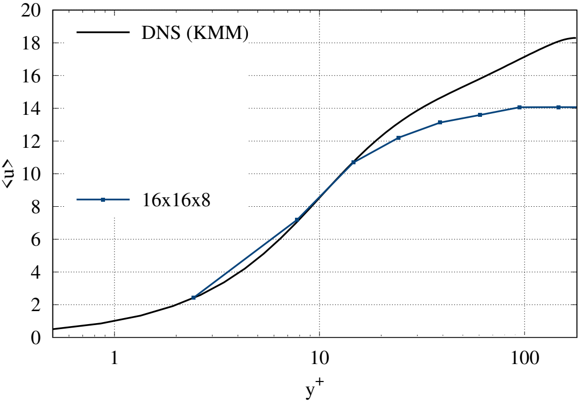

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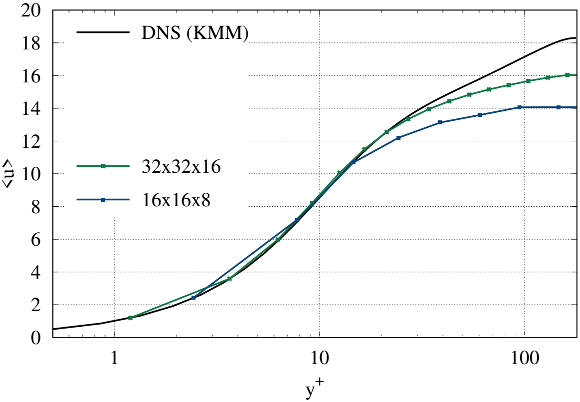

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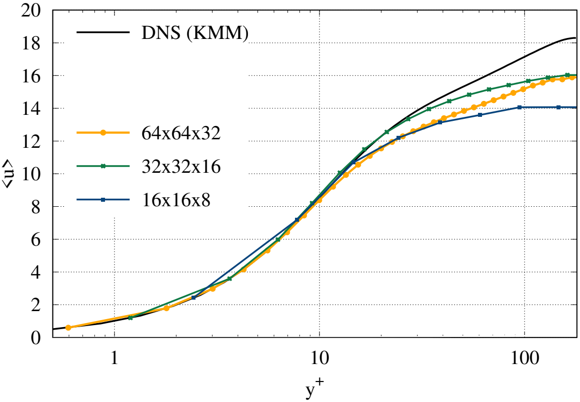

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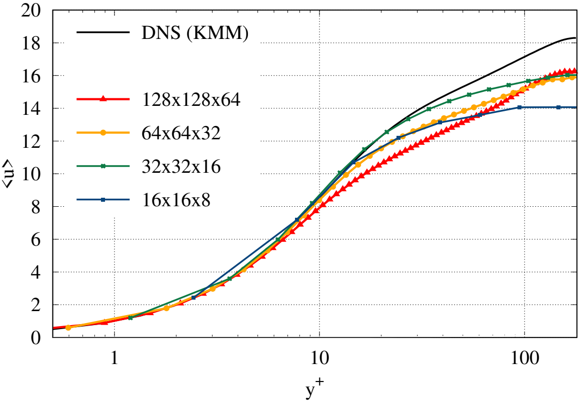

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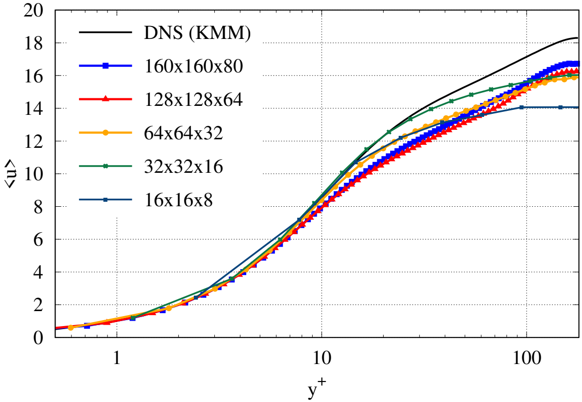

Dispersion errors analysis

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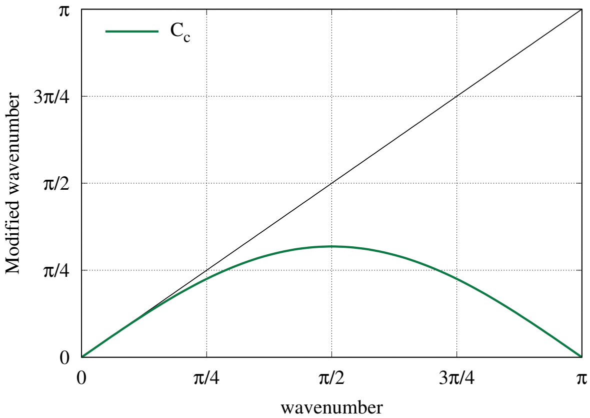

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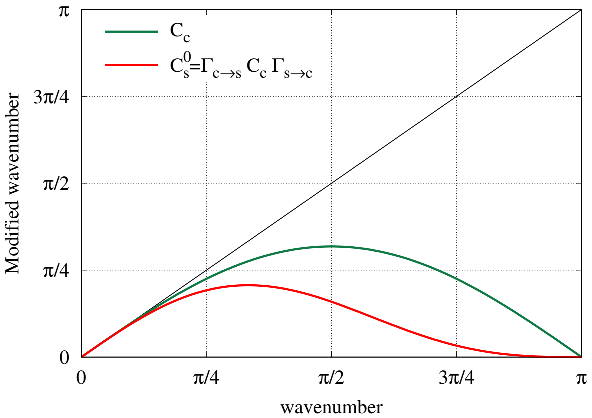

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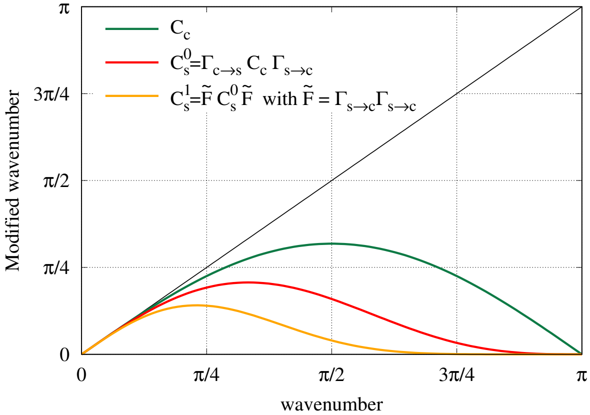

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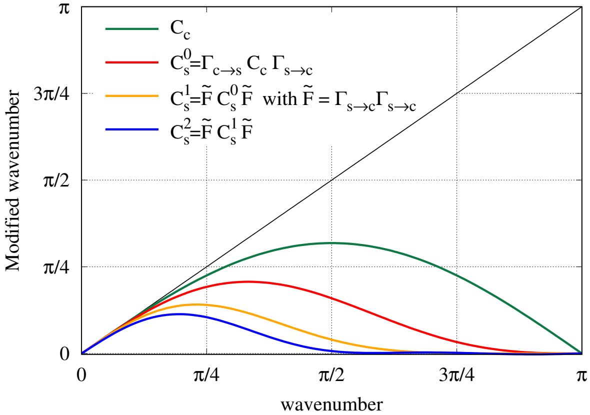

Results for a turbulent channel flow at $Re_\tau = 180$

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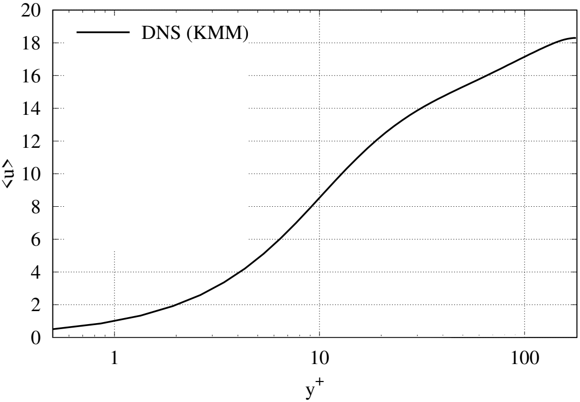

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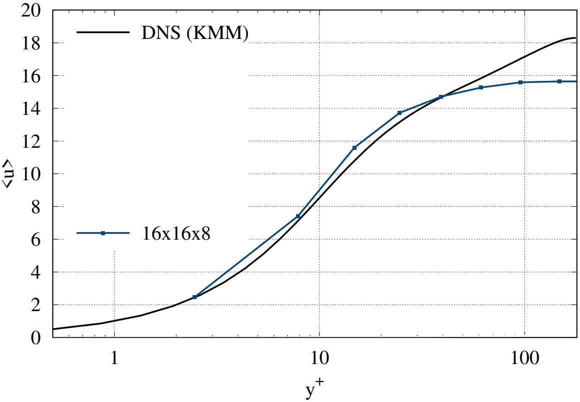

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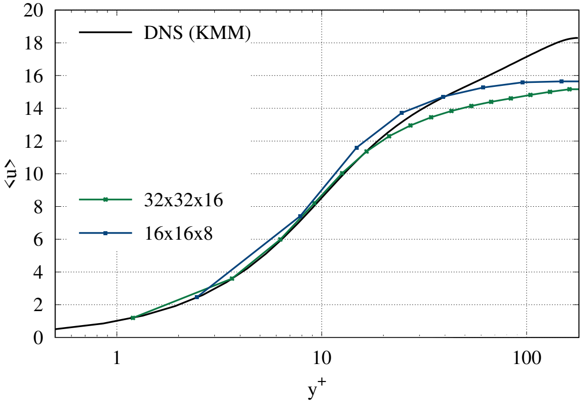

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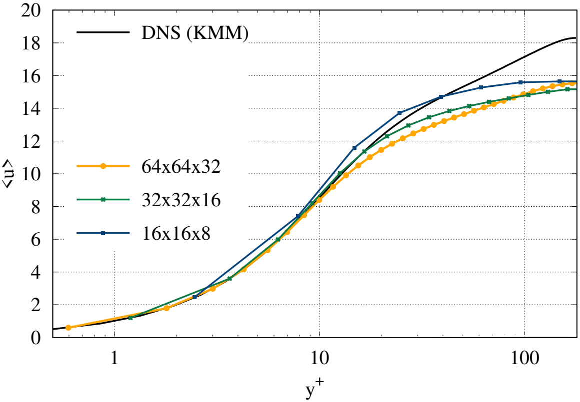

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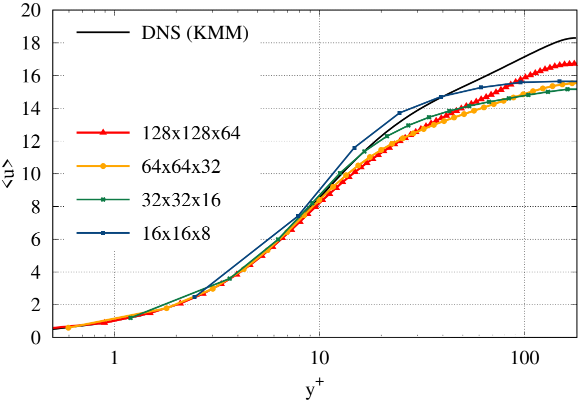

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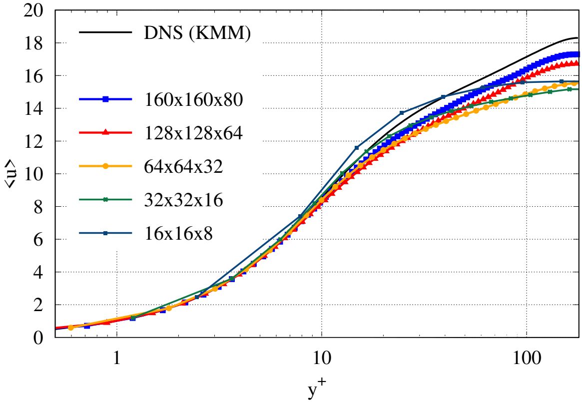

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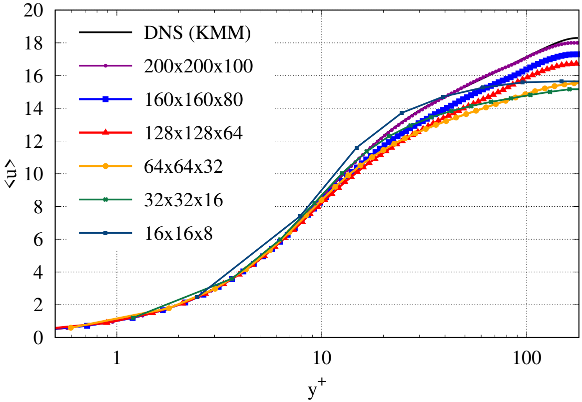
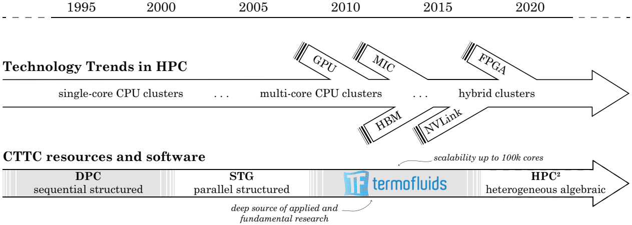


Algebra-based approach naturally leads to portability

Research question #2:

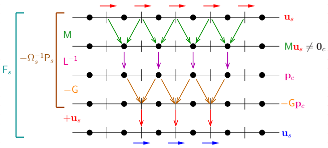
- How can we develop **portable** and **efficient** CFD codes for large-scale simulations on modern supercomputers?

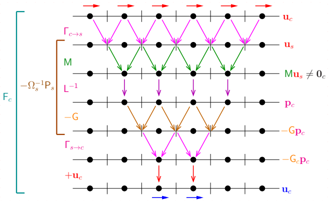


HPC^2: portable, algebra-based framework for heterogeneous computing is being developed. Traditional stencil-based data and sweeps are replaced by algebraic structures (sparse matrices and vectors) and kernels. NUMA-aware execution strategies for CFD are presented in this conference[16].

[16] X.Álvarez, A.Gorobets, F.X.Trias, A.Oliva. *NUMA-aware strategies for the efficient execution of CFD simulations on CPU supercomputers* **ParCFD2021**. Don't miss it!

Algebra-based approach naturally leads to portability, to simple and analyzable formulations

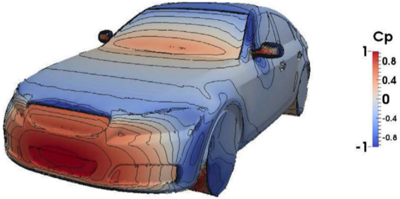


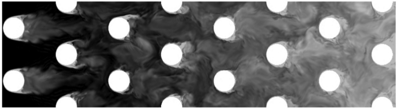


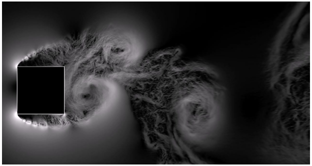
Collocated: $$\boldsymbol{u}_c^{n+1} = \underbrace{\left(I_c - \Gamma_{s\to c}\Omega_s^{-1}\mathsf{P}_s\Gamma_{c\to s}\right)}_{\mathsf{F}_c} \overbrace{\underbrace{\left[I_c + \partial_t^c\right]}_{\mathsf{T}_c} \boldsymbol{u}_c^n}^{\boldsymbol{u}_c^p} = \underbrace{\mathsf{F}_c\mathsf{T}_c}_{NS_c} \boldsymbol{u}_c^n$$

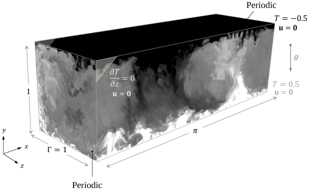
Staggered: $$\boldsymbol{u}_s^{n+1} = \underbrace{\left(I_s - \Omega_s^{-1}\mathsf{P}_s\right)}_{\mathsf{F}_s} \overbrace{\underbrace{\left[I_s + \Gamma_{c\to s}\partial_t^c\Gamma_{s\to c}\right]}_{\mathsf{T}_s} \boldsymbol{u}_s}^{\boldsymbol{u}_s^p}$$

Algebra-based approach naturally leads to portability, to simple and analyzable formulations and opens the door to new strategies[17] to improve its perfomance...









[17] A.Alsalti, X.Álvarez, F.X.Trias, A.Gorobets, A.Oliva. *A highly portable heterogeneous implementation of a Poisson solver for flows with one periodic direction* **ParCFD2021**. Don't miss it!

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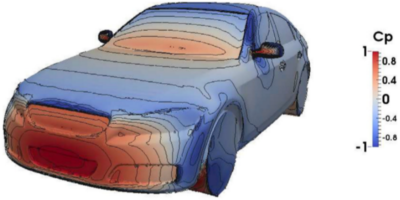
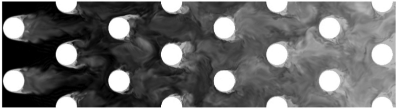
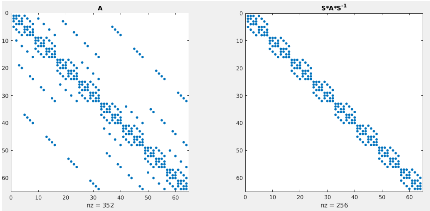




$$\hat{\mathsf{L}} = \mathsf{S}\mathsf{L}\mathsf{S}^{-1} = \mathsf{I} \otimes \mathsf{L}_{inn} + diag(\boldsymbol{d})$$

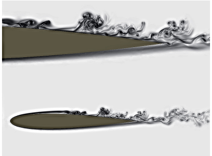


SpMMV can be used $\Longrightarrow$ **higher AI**

[17] A.Alsalti, X.Álvarez, F.X.Trias, A.Gorobets, A.Oliva. *A highly portable heterogeneous implementation of a Poisson solver for flows with one periodic direction* **ParCFD2021**. Don't miss it!

Concluding remarks

- **Preserving symmetries** either using staggered or collocated formulations is the key point for **reliable LES/DNS** simulations.



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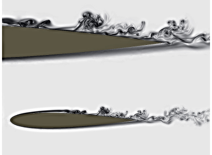
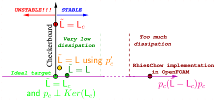
- **Preserving symmetries** either using staggered or collocated formulations is the key point for **reliable LES/DNS** simulations.
- Main drawback of **collocated** formulations: you either have **checkerboard** or some (small) amount of **artificial dissipation** due to pressure term.





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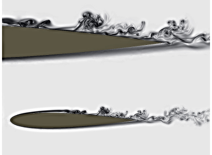
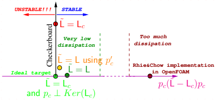
- Despite this, the CFD community have generally adopted collocated formulations due to the inherent **difficulties** to formulate a simple and robust **staggered discretization of momentum**.
  $\Longrightarrow$ A potential solution has been presented here...





- STAR-CCM+
- ANSYS-FLUENT
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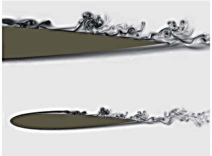

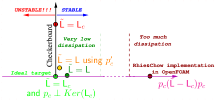

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On-going research:

- Complete the analysis for higher Re_τ
- Test for complex geometries using unstructured meshes

Thank you for your virtual attendance