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Pressure-velocity coupling on unstructured collocated grids: reconciling stability and energy-conservation

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Abstract – In this work, an energy-preserving unconditionally stable fractional step method on collocated unstructured grids is presented. Its formulation is based on preserving the underlying symmetries of the differential operators. This formulation was proven to be unconditionally stable even for highly distorted meshes [1, 2]. Conservation of (global) kinetic energy is also a key feature in simulations. Within this context, a canonical case is tested, a differentially heated cavity, in order to show that the (artificial) kinetic energy error introduced by the pressure is negative and small compared with the physical dissipation.

1. Introduction

General CFD codes such as OpenFOAM or ANSYS-Fluent use a finite-volume (stencil) discretization over unstructured meshes and a collocated formulation to solve the Navier-Stokes equations. The stencil formulations solve the discretised equations using an algorithm that calculates the desired values cell by cell. Alternatively, algebraic formulations keep them in matrix-vector form, and use these matrices and vectors to calculate the desired quantities.

A collocated fully-conservative algebraic symmetry-preserving formulation of incompressible Navier-Stokes equations was proposed by Trias et al.[3]. Assuming n control volumes and m faces:

$$\Omega\frac{d\mathbf{u}_c}{dt}+\mathbf{C}(\mathbf{u}_s)\mathbf{u}_c = \mathbf{D}\mathbf{u}_c-\Omega\mathbf{G}_c p_c, \quad (1)$$

$$\mathbf{M}\mathbf{u}_s = \mathbf{0}_c, \quad (2)$$

where $\mathbf{u}_c\in\mathbb{R}^{3n}$ and $\mathbf{p}_c\in\mathbb{R}^n$ are the cell-centered velocity and the cell-centered pressure, respectively. The face-centered quantities, such as $\mathbf{u}_s\in\mathbb{R}^m$ are related to the cell-centered quantities via an interpolation operator $\Gamma_{c\to s}\in\mathbb{R}^{m\times 3n}$:

$$\mathbf{u}_s=\Gamma_{c\to s}\mathbf{u}_c. \quad (3)$$

Finally, $\Omega\in\mathbb{R}^{3n\times 3n}$ is a diagonal matrix containing the cell volumes, $\mathbf{C}(\mathbf{u}_s)\in\mathbb{R}^{3n\times 3n}$ is the discrete convective operator, $\mathbf{D}\in\mathbb{R}^{3n\times 3n}$ is the discrete diffusive operator, $\mathbf{G}_c\in\mathbb{R}^{3n\times n}$ is the cell-to-cell discrete gradient operator and $\mathbf{M}\in\mathbb{R}^{n\times m}$ is the face-to-cell discrete divergence operator. After applying the Fractional Step Method (FSM) to the Navier-Stokes equations, the velocity correction reads:

$$\mathbf{u}_c^{n+1}=\mathbf{u}_c^{p}-\Gamma_{s\to c}\mathbf{G}p_c^{n+1}, \quad (4)$$

where $\Gamma_{s\to c}\in\mathbb{R}^{3n\times m}$ is the face-to-cell interpolator, which is related to the cell-to-face interpolator via the volume matrices $\Gamma_{s\to c}=\Omega^{-1}\Gamma_{c\to s}\Omega_s$, and $\mathbf{G}\in\mathbb{R}^{m\times n}$ is the cell-to-face gradient operator.

Only three discrete geometrical operators are needed to formulate these equations: the cell-centered and staggered control volumes (diagonal matrices), Ω_c and Ω_s, the face normal vectors, N_s; plus two non-geometrical ones: the scalar cell-to-face interpolation, $\Pi_{c\to s}$ and the cell-to-face divergence operator, $\mathbf{M}$. For more details of these operators and its construction, the reader is referred to [3]. Due to its simplicity, these operators can be easily built in existing codes, such as OpenFOAM [4]. A fully-portable algebra-based HPC^2 framework can be found in [5].

2. An energy-preserving unconditionally stable FSM

From our point of view, the physical structure of the equations is only respected when the symmetries of these differential operators is preserved. For instance, constructing $\mathbf{G} = -\Omega_s \mathbf{M}^T$ is necessary to preserve kinetic energy [3], but it is also mimicking the symmetries of the continuous level operators [6].

The turbulence phenomenon is caused by a balance between convective transport and diffusive dissipation. The discrete forms of these two physical processes are defined by $\mathbf{C}(\mathbf{u}_s)$ and $\mathbf{D}$, respectively. At the continuous level, the convective operator is skew-symmetric, while the diffusive operator is symmetric and negative-definite. If we keep these properties at the discrete level (namely $\mathbf{C}(\mathbf{u}_s)$ being a skew-symmetric matrix, $\mathbf{D}$ being a symmetric negative-definite matrix and $\mathbf{G} = -\Omega_s \mathbf{M}^T$), the discrete convective operator will transport energy from resolved motion scales to others without dissipating energy.

The utility of an algebraic formulation can be found, as an example, in [1, 2]. In these works, the matrix-vector formulation is employed to investigate the stability of the solution in terms of the pressure gradient interpolation in collocated frameworks. To do so, the eigenvalues of $\mathbf{L} - \mathbf{L}_c$ were deeply studied ($\mathbf{L} = \mathbf{MG} \in \mathbb{R}^{n\times n}$ is the compact Laplacian operator whereas $\mathbf{L}_c = \mathbf{M}\Gamma_{c\to s}\Gamma_{s\to c}\mathbf{G} \in \mathbb{R}^{n\times n}$ is the collocated wide-stencil Laplacian operator), and the cell-to-face interpolation that leads to an unconditionally stable FSM turned out to be:

$$\Pi_{c\to s} = \Delta_s^{-1}\Delta_{sc}^T \in \mathbb{R}^{m\times n}, \tag{5}$$

where $\Delta_s \in \mathbb{R}^{m\times m}$ is a diagonal matrix containing the projected distances between two adjacent control volumes, and $\Delta_{sc} \in \mathbb{R}^{n\times m}$ is a matrix containing the projected distance between a cell node and its corresponding face. For details, the reader is encouraged to consult[1, 2].

3. Conservation of global kinetic energy

The global discrete kinetic energy temporal evolution equation can be obtained by left-multiplying Eq. (1) by $\mathbf{u}_c^T$ and adding it with its transpose. Taking into account that the convective operator should be skew-symmetric:

$$\frac{d}{dt}||\mathbf{u}_c||^2 \;=\; \mathbf{u}_c^T(\mathbf{D}+\mathbf{D}^T)\mathbf{u}_c - \mathbf{u}_c^T\Omega\mathbf{G}_c p_c - p_c^T\mathbf{G}_c^T\Omega^T\mathbf{u}_c. \tag{6}$$

The pressure error contribution introduced to the discrete kinetic energy equation is zero for symmetry-preserving staggered formulations, due to the fact that $\mathbf{G} = -\Omega_s\mathbf{M}^T$, and the incompressibility constraint $\mathbf{Mu}_s = 0$. However, in collocated formulations $\mathbf{M}_c\mathbf{u}_c \approx 0$, but not strictly zero.

In collocated framework, the (artificial) kinetic energy added is given by:

$$-\mathbf{p}_c^T G_c^T \Omega^T \mathbf{u}_c = \mathbf{p}_c^T(\mathbf{L}-\mathbf{L}_c)\mathbf{p}_c \tag{7}$$

The interpolation shown in Eq. (5) assures that $\mathbf{L} - \mathbf{L}_c$ is negative definite [1, 2], so the contribution of the (artificial) kinetic energy added by the pressure term is dissipative, ensuring the stability of the simulation. However, it is important to quantify it.

4. Stability of the method

The stability of the method was confirmed using numerical tests that apply extremely coarse and very poor quality meshes. Figures 1 and 2 show an example of an air-filled ($Pr = 0.71$) differentially heated cavity with aspect ratio 2 at Rayleigh number (based on the cavity height) of 10^6, respecting operator symmetries and interpolating the pressure gradient using Eq. (5).

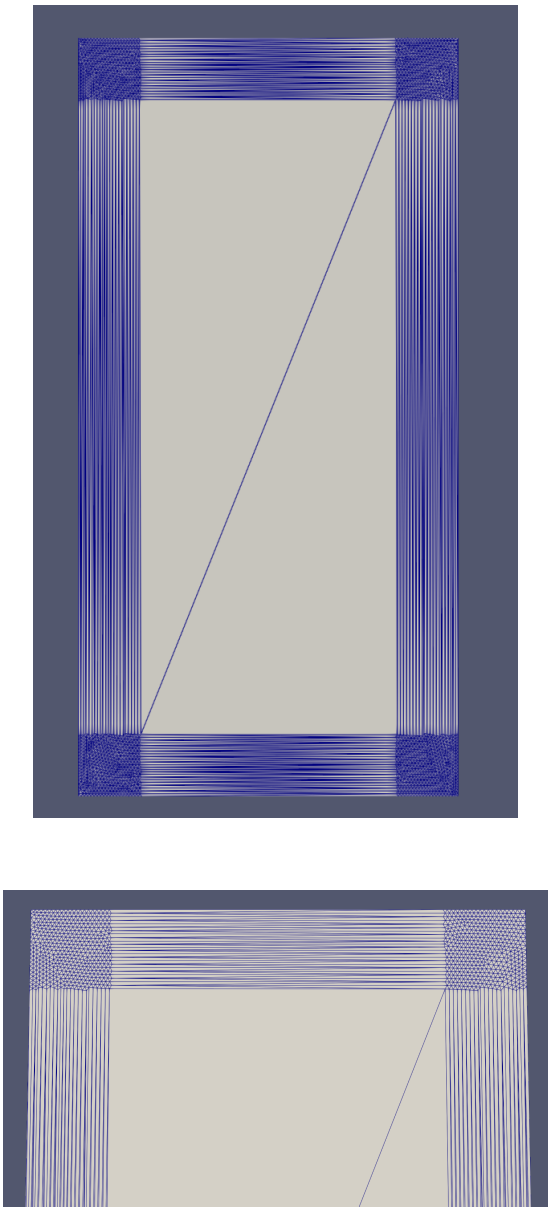

Figure 1: (Top) Test mesh used to check the stability of the method. (Bottom) Zoom at the top part of the mesh.

As seen in Fig. 2, the method is stable, producing results without blowing up the simulation. The accuracy will be low, as we would anticipate from such a low-quality mesh. It is

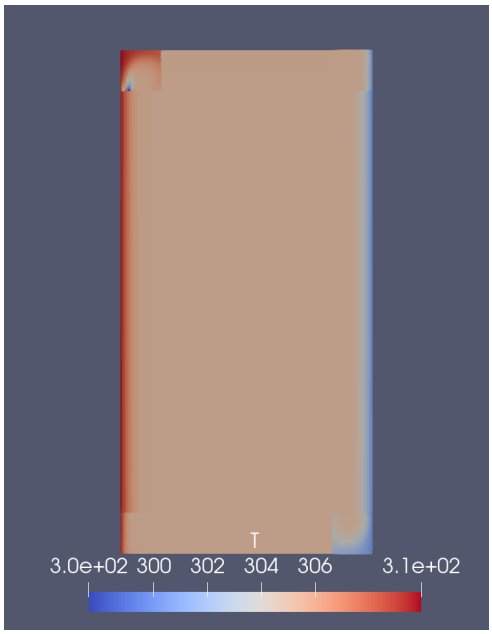

Figure 2: Temperature distribution obtained for $Ra = 10^6$ using the mesh displayed in Figure 1.

worth noting that using different interpolations for the pressure gradient, such as $\frac{1}{2}$ weights will immediately blow up the simulation.

Once we assured the stability of the method, the energy budgets will be computed, in order to quantify the (artificial) contribution of the pressure term to the discrete kinetic energy, as it is done in [3].

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