Progress on eddy-viscosity models for LES: new differential operators and discretization methods

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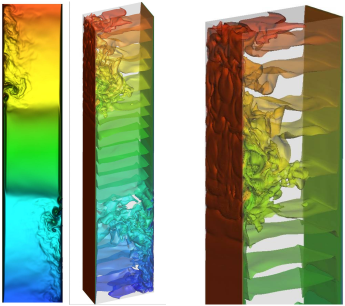
Contents

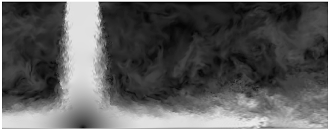
DNS of turbulent incompressible flows

Main features of the DNS code:

- Structured staggered grids
- High-order symmetry-preserving schemes
- Fully-explicit second-order time-integration method
- Poisson solver for 2.5D problems: FFT + PCG
- Hybrid MPI+OpenMP parallelization
- OpenCL-based extension for its use on GPGPU

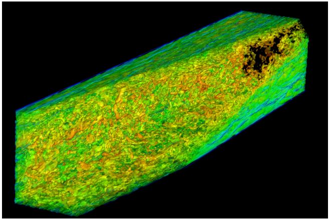


Air-filled differentially heated cavity at $Ra = 10^{11}$ (111M grid points)

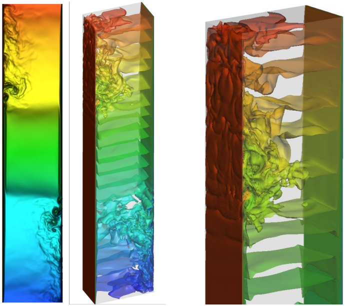


Plane impingement jet at $Re = 20000$ (102M grid points)

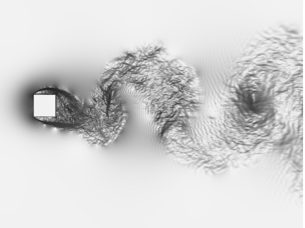
DNS of turbulent incompressible flows



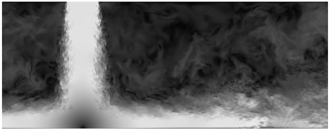
Turbulent square duct at $Re_\tau = 1200$ (172M grid points)



Air-filled differentially heated cavity at $Ra = 10^{11}$ (111M grid points)



Square cylinder at $Re = 22000$ (300M grid points)



Plane impingement jet at $Re = 20000$ (102M grid points)

Governing equations

Incompressible Navier-Stokes equations:

$$\begin{aligned} \nabla \cdot u &= 0 \\ \partial_t u + \mathcal{C}(u, u) &= \mathcal{D}u - \nabla p \end{aligned}$$

where the **nonlinear convective** term is given by

$$\mathcal{C}(u, \phi) = (u \cdot \nabla)\phi$$

and the **linear dissipative** term is given by

$$\mathcal{D}\phi = \nu \Delta \phi$$

Stopping the vortex-stretching[1]

Taking the curl of momentum equation the **vorticity transport equation** follows

$$\partial_t \omega + \mathcal{C}(u, \omega) = \mathcal{C}(\omega, u) + \mathcal{D}(\omega)$$

Let us now consider an arbitrary part of the flow domain, Ω, with **periodic boundary conditions**. Then, taking the L^2 innerproduct with $\omega = \nabla \times u$ leads to the **enstrophy equation**

$$\frac{1}{2}\frac{d}{dt}(\omega, \omega) = (\omega, \mathcal{C}(\omega, u)) - \nu(\nabla\omega, \nabla\omega)$$

where $(a, b) = \int_\Omega a \cdot b d\Omega$. Unless, the grid is fine enough convection dominates diffusion (in a discrete sense)

$$(\omega, \mathcal{C}(\omega, u)) > \nu(\nabla\omega, \nabla\omega)$$

[1] F.X. Trias *et al.* **Computers&Fluids**, 39:1815-1831, 2010

Stopping the vortex-stretching

The vortex-stretching term can be expressed in terms of the invariant $R = -1/3tr(S^3) = -det(S)$

$$(\omega, \mathcal{C}(\omega, u)) = 4\int_\Omega R d\Omega$$

Then, recalling that $\nabla \times \omega = \nabla(\nabla \cdot u) - \Delta u$ and the boundary contribution vanishes *, the diffusive term is given by the $L^2(\Omega)$-norm of Δu

$$\begin{aligned}(\nabla\omega, \nabla\omega) &\stackrel{*}{=} -(\omega, \Delta\omega) = (\omega, \nabla \times \nabla \times \omega) \\ &\stackrel{*}{=} (\nabla \times \omega, \nabla \times \omega) = (\Delta u, \Delta u) = \|\Delta u\|^2\end{aligned}$$

Stopping the vortex-stretching

The **overall damping** introduced by a model should be given by

$$H^{\Omega} = \min\left\{ \frac{\nu \|\Delta u\|^2}{4|\widetilde{R}|}, 1 \right\}$$

where $\widetilde{R} = \int_{\Omega} R d\Omega$.

Notice that any model based on this ratio automatically **switches off** for:

- Laminar flows ($R \to 0$)
- 2D flows ($\lambda_2 = 0 \longrightarrow R = 0$)
- In the wall (near-wall behavior is given by $R \propto y^1$ and $\|\Delta u\|^2 \propto y^0$)

Stopping the vortex-stretching

The **overall damping** introduced by a model should be given by

$$H^{\Omega} = \min \left\{ \frac{\nu \|\Delta u\|^2}{4|\widetilde{R}|}, 1 \right\}$$

One possible solution would consist on an **eddy-viscosity** type LES model:

$$\nu_t \approx \frac{4|\widetilde{R}|}{\|\Delta u\|^2}$$

Taking $\|\Delta u\|^2 \leq -\lambda_{\Delta}(\omega, \omega) = 4\lambda_{\Delta}\widetilde{Q}$, it becomes the eddy-viscosity model[2] based on the invariants $R = -1/3tr(S^3) = -det(S)$ and $Q = -1/2tr(S^2)$.

$\lambda_{\Delta} < 0$ is the largest (smallest in absolute value) non-zero eigenvalue of Laplacian operator Δ on Ω. In a periodic box of size h, $\lambda_{\Delta} = -(\pi/h)^2$.

[2] Roel Verstappen, **Journal of Scientific Computing**, 49:94-110, 2011

Stopping the vortex-stretching

The **overall damping** introduced by a model should be given by

$$H^{\Omega} = \min\left\{ \frac{\nu \|\Delta u\|^2}{4|\widetilde{R}|}, 1 \right\}$$

Alternatively, **regularizations** of the non-linear convective term results into a damping of vortex-stretching term, *i.e.* $f^{reg}|\tilde{R}|$ (where $0 < f \leq 1$)

$$f^{reg} \approx \min\left\{ \frac{\nu \|\Delta u\|}{4|\tilde{R}|}, 1 \right\}$$

Or a combination of both?

Towards a simple LES model

Hence, a **new eddy-viscosity** model for LES

$$\partial_t \overline{u} + \mathcal{C}(\overline{u}, \overline{u}) = \mathcal{D}\overline{u} - \nabla \overline{p} - \nabla \cdot \tau(\overline{u}) \quad ; \qquad \nabla \cdot \overline{u} = 0$$

$$\tau \ (\overline{u}) = -2\nu_t S(\overline{u})$$

has been derived from the criterion that vortex-stretching mechanism must stop at the smallest grid scale

$$\boxed{\nu_t \approx \frac{4|\widetilde{R}|}{\|\Delta \overline{u}\|^2}}$$

And what about the implementation?

- No problems with $4|\widetilde{R}|$ and $\|\Delta \overline{u}\|^2$.
- But, what about the **discretization** of $\nabla \cdot \tau(\overline{u})$?

Discretization of $2\nabla\cdot(\nu_t S(u))$: a new simple approach[3]

$$\partial_t u + \mathcal{C}(u,u) = \mathcal{D}u - \nabla p \quad + 2\nabla\cdot(\nu_t S(u)), \qquad \nabla\cdot u = 0$$

$$\Omega_s \frac{du_s}{dt} + \mathsf{C}(u_s)\,u_s = \mathsf{D}u_s + \mathsf{M}^T p_c + \quad ?????? \quad , \qquad \mathsf{M}u_s = 0_c$$

where $2\nabla\cdot(\nu_t S(u)) = \nabla\cdot(\nu_t\nabla u) + \nabla\cdot(\nu_t\nabla u^T)$.

$$\begin{aligned}\nabla\cdot(\nu_t\nabla u^T) &= \nabla(\nabla\cdot(\nu_t u)) - \nabla\cdot(u\otimes\nabla\nu_t)\\ &= \nabla(\nabla\cdot(\nu_t u)) - \mathcal{C}(u,\nabla\nu_t)\end{aligned}$$

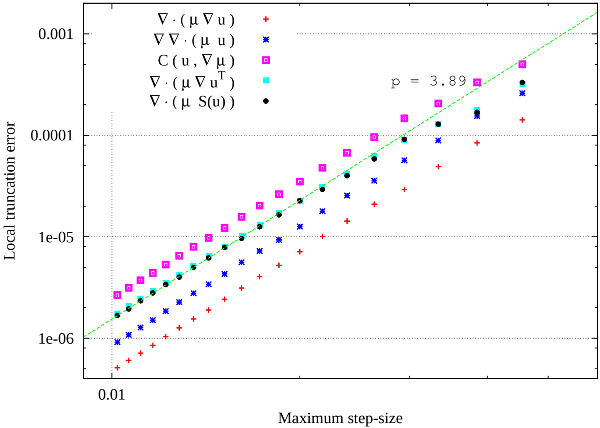
$$\boxed{\underbrace{-\mathsf{M}^T\Omega_c^{-1}\mathsf{M}\tilde{u}_s}_{\approx\nabla(\nabla\cdot(\nu_t u))} - \underbrace{\mathsf{C}(u_s)(-\Omega_s^{-1}\mathsf{M}^T\nu_{t,c})}_{\approx\mathcal{C}(u,\nabla\nu_t)}}$$

where $[\tilde{u}_s]_f = [\nu_{t,s}]_f[u_s]_f$. **Straightforward implementation!!!**

[3] F.X.Trias *et al. A simple approach to discretize the viscous term with spatially varying (eddy-)viscosity* **Journal of Computational Physics**, 253:405-417, 2013

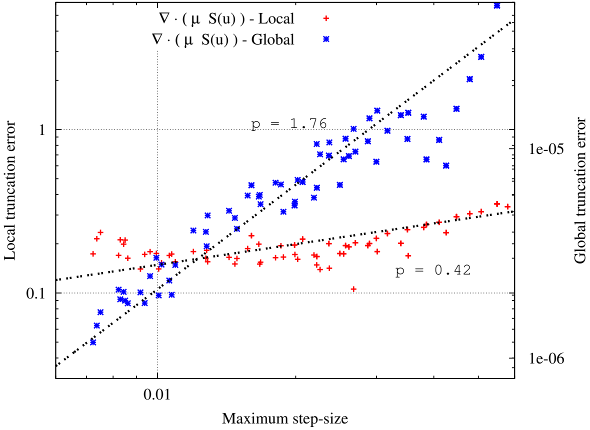
Discretization of $2\nabla \cdot (\nu_t S(u))$: a new simple approach

4^{th}-order FVM on a staggered Cartesian grid



Discretization of $2\nabla \cdot (\nu_t S(u))$: a new simple approach

2^{th}-order FVM on a collocated unstructured grid



Discretization of $2\nabla \cdot (\nu_t S(u))$: a new simple approach

Let's make it even easier...

$$\nabla \cdot (\nu_t \nabla u^T) = \nabla(\nabla \cdot (\nu_t u)) - \mathcal{C}(u, \nabla \nu_t)$$

$$\boxed{\underbrace{-\mathsf{M}^T \Omega_c^{-1} \mathsf{M} \tilde{u}_s}_{\approx \nabla(\nabla \cdot (\nu_t u))} - \underbrace{\mathsf{C}(u_s)(-\Omega_s^{-1} \mathsf{M}^T \nu_{t,c})}_{\approx \mathcal{C}(u, \nabla \nu_t)}}$$

Since $\nabla(\nabla \cdot (\nu_t u))$ is a gradient of a scalar field, this term can be **absorbed into the pressure**, $\pi = p - \nabla \cdot (\nu_t u)$.

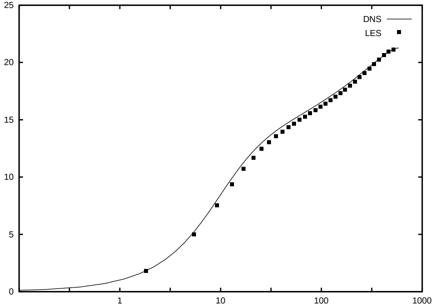
Therefore, the only term that needs to be discretized is

$$\boxed{-\underbrace{\mathsf{C}(u_s)(-\Omega_s^{-1} \mathsf{M}^T \nu_{t,c})}_{\approx \mathcal{C}(u, \nabla \nu_t)}}$$

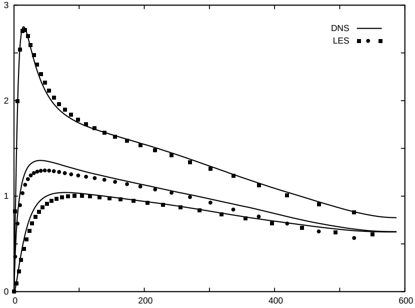
Preliminary results

Turbulent channel flow

$Re_\tau = 590$ DNS Moser et al. LES 64^3



mean velocity

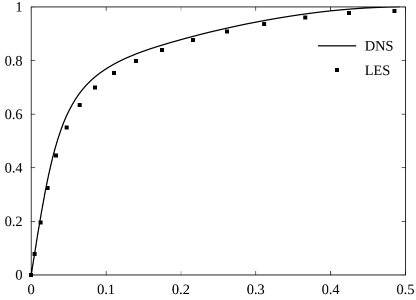


rms fluctuations

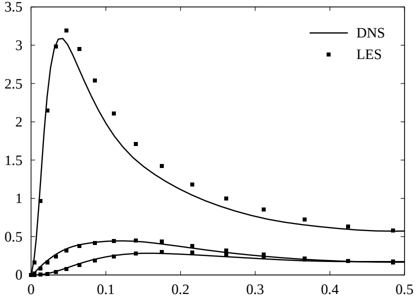
Preliminary results

Turbulent square duct

$Re_\tau = 300$ LES $64 \times 32 \times 32$



mean velocity



rms fluctuations

Conclusions and Future Research

- The ratio between the invariant R and the (total) dissipation provides a proper differential operator for turbulence models.
- Based on this, **a new eddy-viscosity type** LES models has been derived.
- A simple **new approach to discretize** the viscous term for **eddy-viscosity models** has been proposed.

- Test the performance of new eddy-viscosity type LES for other configurations.
- Try to properly combine regularization modeling and LES.

Thank you for your attention

Further reading

- Roel Verstappen, *"When does eddy viscosity damp subfilter scales sufficiently?"*, Journal of Scientific Computing, 49 (1): 94-110, 2011
- F.X.Trias, R.W.C.P.Verstappen, A.Gorobets, M.Soria, A.Oliva, *"Parameter-free symmetry-preserving regularization modeling of a turbulent differentially heated cavity"*, Computers & Fluids, 39:1815-1831, 2010.
- F.X.Trias, A.Gorobets, A.Oliva, *A simple approach to discretize the viscous term with spatially varying (eddy-)viscosity*, Journal of Computational Physics, 253:405-417, 2013.